Problem 5 - Solution

Hodograph solution

The hodograph of the satellite's elliptical orbit is a circle. The orbit's eccentricity vector is conserved:

$$\mathbf{e} = \frac{\mathbf{p} \times \mathcal{L}}{GMm^2} - \hat{\mathbf{r}} = \frac{\mathbf{v} \times \mathcal{L}}{GMm} - \hat{\mathbf{r}}$$

Proof. Deriving the first term with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \times \mathcal{L}) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \times \mathcal{L} + \mathbf{p} \times \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \mathbf{F} \times \mathcal{L}$$

The second term vanishes because the gravitational force is a central force, i.e. its torque is identically zero.

$$\mathbf{F} \times (\mathbf{r} \times \mathbf{p}) = m\left[\left(\mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)\mathbf{r} - (\mathbf{F} \cdot \mathbf{r})\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right]$$

Upon expressing the gravitational force vector as follows:

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}$$

Note that $\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}^2) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = r\frac{\mathrm{d}r}{\mathrm{d}t}$; hence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{p} \times L}{GMm^2}\right) = -\left[\frac{1}{r^3}\left(\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)\mathbf{r} - \frac{1}{r}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right] =$$

$$= -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\mathbf{r} + \frac{1}{r}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}})$$

Notice that the obtained derivative is equal to the one being subtracted in the defined eccentricity vector, therefore:

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}} - \hat{\mathbf{r}}) = \mathbf{0} \qquad \square$$

We express $\hat{\mathbf{r}}$ from the expression of $\mathbf{e}$ and take its dot product with itself to obtain an equation of a circle.

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \left(\frac{\mathbf{v} \times \mathcal{L}}{GMm} - \mathbf{e}\right) \cdot \left(\frac{\mathbf{v} \times \mathcal{L}}{GMm} - \mathbf{e}\right) =$$

$$= \frac{\mathbf{v}^2\mathcal{L}^2}{(GMm)^2} - 2\left(\frac{\mathbf{v} \times \mathcal{L}}{GMm}\right) \cdot \mathbf{e} + \mathbf{e}^2$$

Let the x-axis lie along the ellipse's major axis, and the z-axis normal to the orbital plane. Observe the middle term:

$$(\mathbf{v} \times \mathcal{L}) \cdot \mathbf{e} = -(\mathbf{v} \times \mathbf{e}) \cdot \mathcal{L}$$

$$\therefore \quad v_x^2 + \left(v_y - \frac{GMme}{L}\right)^2 = \left(\frac{GMm}{L}\right)^2$$

We easily identify the circle's radius as $R = GMm/L$ and its centre as $S(0, GMme/L)$. We introduce the quantity called *offset* as the distance from the origin and the circle's centre S such that $d = |OS| = GMme/L$. Note that $d = eR$.

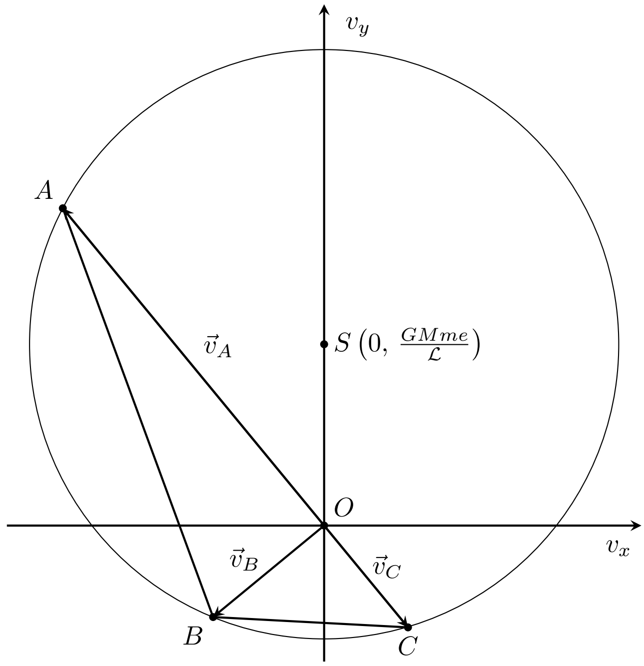

Figure 1: Satellite's hodograph.

Consider the triangle formed by the given velocities. We may calculate its surface area in two manners:

$$S_{\triangle ABC} = \frac{(v_1 + v_3)\sqrt{v_1^2 + v_2^2}\sqrt{v_2^2 + v_3^2}}{4R} = \frac{(v_1 + v_3)v_2}{2}$$

$$\therefore \quad R = \frac{\sqrt{v_1^2 + v_2^2}\sqrt{v_2^2 + v_3^2}}{2v_2} \tag{1}$$

Applying the intersecting chord theorem (proof below) on the chord going through O parallel to velocities v_1 and v_3:

$$v_1 v_3 = (R + d)(R - d) \implies e^2 = \frac{d^2}{R^2} = 1 - \frac{v_1 v_3}{R^2} \tag{2}$$

Upon inserting equation (1) into equation (2), we find the final answer for the orbit's eccentricity:

$$e = \sqrt{1 - \frac{4v_1 v_3 v_2^2}{(v_1^2 + v_2^2)(v_2^2 + v_3^2)}}$$

In the given special case $v_1 = 1\,\mathrm{km/s}$, $v_2 = 2\,\mathrm{km/s}$, $v_3 = 3\,\mathrm{km/s}$, the eccentricity numerically evaluates to:

$$e = \sqrt{\frac{17}{65}}$$

Proof. Intersecting chords theorem. Using points A, B and C from figure 1 without loss of generality, we draw point D as the intersection of line BO with the circle. Consider the angles of triangles $\triangle AOD$ and $\triangle BOC$:

$$|\angle ADO| = |\angle BCO| \quad \text{(inscribed angles over } \overline{AB}\text{)}$$

$$|\angle DAO| = |\angle CBO| \quad \text{(inscribed angles over } \overline{CD}\text{)}$$

$$|\angle AOD| = |\angle BOC| \quad \text{(opposing angles)}$$

$$\therefore \quad \triangle AOD \sim \triangle BOC$$

$$\frac{|AO|}{|OD|} = \frac{|BO|}{|OC|} \iff |AO| \cdot |OC| = |BO| \cdot |OD| \qquad \square$$

Alternative solution

We find the relation between the angles of true anomaly θ and the eccentric anomaly ϕ measured from the major axis.

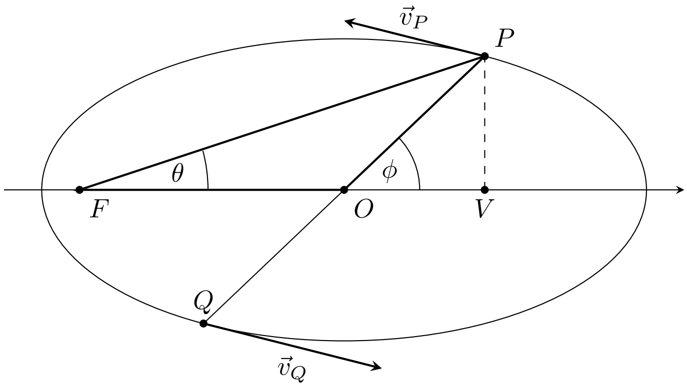

Figure 2: Satellite's elliptical orbit.

For a point P on the orbit whose projection onto the major axis of an orbit with centre O is V, $|OV| = a\cos\phi$. Let F be the ellipse's focus, so $|OV| = |OF| + |FV| = ae + r\cos\theta$. Combining the latter with the formula for the orbital radius:

$$r = \frac{a(1-e^2)}{1-\cos\theta} \implies \cos\theta = \frac{\cos\phi - e}{1 - e\cos\phi}$$

Proof. Orbital radius formula. We express the dot product of the satellite's angular momentum with itself using the previously discussed eccentricity vector:

$$\mathcal{L}^2 = \mathcal{L}\cdot(\mathbf{r}\times\mathbf{p}) = \mathbf{r}\cdot(\mathbf{p}\times\mathcal{L}) = GMm^2 r(1+e\cos\theta)$$

$$\therefore \quad r = \frac{\mathcal{L}^2}{GMm^2}\frac{1}{1+e\cos\theta}$$

We take the dot product of the eccentricity vector with itself:

$$\mathbf{e}^2 = \frac{\mathbf{p}^2\mathcal{L}^2}{(GMm^2)^2} + 1 = \frac{2mE\mathcal{L}^2}{(GMm^2)^2} + 1 = -\frac{L^2}{GMm^2 a} + 1$$

$$\therefore \quad \frac{\mathcal{L}^2}{GMm^2} = a(1-e^2)$$

The vis-viva equation with the eccentric anomaly ϕ as its argument then becomes:

$$v^2 = \frac{2GM}{a}\frac{1+e\cos\phi}{1-e\cos\phi}$$

Notice that point C is centrally-symmetric to A with respect to the centre O because $\mathbf{v}_A \parallel \mathbf{v}_B$; hence, $\phi_C = \pi + \phi_A$:

$$v_A^2 = \frac{2GM}{a}\frac{1+e\cos\phi_A}{1-e\cos\phi_A} \quad \text{and} \quad v_C^2 = \frac{2GM}{a}\frac{1+e\cos\phi_C}{1-e\cos\phi_C} =$$

$$= \frac{2GM}{a}\frac{1-e\cos\phi_A}{1+e\cos\phi_A} \implies \frac{v_A}{v_C} = \frac{1+e\cos\phi_A}{1-e\cos\phi_A}$$

We find $\cos\phi_A$ from the last expression, and express $\cot\phi_A$ (for reasons yet to be revealed) to find:

$$\cos\phi_A = \frac{1}{e}\frac{v_A - v_C}{v_A + v_C} \implies \cot\phi_A = \pm\frac{1}{\sqrt{e^2\left(\frac{v_A+v_C}{v_A-v_C}\right)^2 - 1}}$$

Computing v_A/v_B while recognising v_A/v_C to find $\tan\phi_B$:

$$\left(\frac{v_A}{v_B}\right)^2 = \frac{\frac{1+e\cos\phi_A}{1-e\cos\phi_A}}{\frac{1+e\cos\phi_B}{1-e\cos\phi_B}} \implies \cos\phi_B = \frac{1}{e}\frac{v_B^2 - v_Av_C}{v_B^2 + v_Av_C}$$

$$\tan\phi_B = \pm\sqrt{e^2\left(\frac{v_B^2+v_Av_C}{v_B^2-v_Av_C}\right)^2 - 1}$$

For an ellipse given by $x^2/a^2 + y^2/b^2 = 1$, the slope of the tangent at a given point $P(a\cos\phi,\, b\sin\phi)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0 \implies k_P = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_P = -\frac{x}{y}\frac{b^2}{a^2} = -\frac{b}{a}\cot\phi$$

From $b/a = \sqrt{1-e^2}$ and the given condition that $\mathbf{v}_A \perp \mathbf{v}_B$, i.e. the tangents drawn at points A and B are perpendicular:

$$k_Ak_B = -1 \implies (1-e^2)\cot\phi_A\cot\phi_B = -1$$

$$\therefore \quad \tan\phi_B = -(1-e^2)\cot\phi_A$$

It is now clear why we searched for $\cot\phi_A$ and $\tan\phi_B$, but we have to be careful to choose their appropriate sign. Since $\mathbf{v}_A$ and $\mathbf{v}_B$ are perpendicular, one of their slopes must be negative, so we choose one of the two co-tangents to be negative:

$$\sqrt{e^2\left(\frac{v_B^2+v_Av_C}{v_B^2-v_Av_C}\right)^2 - 1} = \frac{1-e^2}{\sqrt{e^2\left(\frac{v_A+v_C}{v_A-v_C}\right)^2 - 1}}$$

It is now only a matter of algebra to find an explicit expression for the orbit's eccentricity. When simplifying the equations, one has to pay close attention to the positive/negative signs under the roots. After all algebraic work, we finally arrive at the following explicit expression for the orbit's eccentricity:

$$e = \sqrt{\frac{\left(\frac{v_A+v_C}{v_A-v_C}\right)^2 + \left(\frac{v_B^2+v_Av_C}{v_B^2-v_Av_C}\right)^2 - 2}{\left(\frac{v_A+v_C}{v_A-v_C}\cdot\frac{v_B^2+v_Av_C}{v_B^2-v_Av_C}\right)^2 - 1}}$$

Upon plugging the given values of $v_A = 1\,\mathrm{km/s}$, $v_B = 2\,\mathrm{km/s}$ and $v_C = 3\,\mathrm{km/s}$, we confirm the answer in our previous solution:

$$e = \sqrt{\frac{17}{65}}$$