# PC 2025 P5 Solution

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## 1 Introduction

This solution will make use of the eccentricity vector and the fact that it is constant, which is defined as:

$$\vec{\varepsilon} = \frac{\vec{v} \times \vec{L}}{GMm} - \hat{r}$$

where $\vec{v}$ is the speed of the sattelite, $\vec{L}$ is the angular momentum, $m$ is its mass, and $M$ is the mass of the planet, and finally, $\hat{r}$ is the unit vector of the position vector from the planet to the sattelite, $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$. Note that the magnitude of the unit vector is, unsurprisingly, 1 and only it's direction changes - this will be crucial later, and that the magnitude of $\vec{\varepsilon}$ is the eccentricity of the orbit we are looking for. Also note that all three of the given velocities are positive. The solution will later transition into a geometric one.

## 2 Transitioning to geometry

### 2.1 Proof of the eccentricity vector being constant

Firstly, note that the angular momentum of the sattelite remains constant. We can rewrite $\vec{L}$ as $mr^2\vec{\omega}$. Also, for a vector that only changes in direction and not length $\vec{a}$, we know that $\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{\omega} \times \vec{a}$. Taking the derivative of the eccentricity vector:

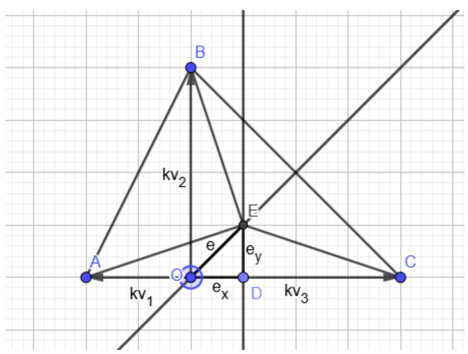
$$\frac{\mathrm{d}\vec{\varepsilon}}{\mathrm{d}t} = \frac{\vec{a} \times (mr^2\vec{\omega})}{GMm} - \vec{\omega} \times \hat{r} = \frac{1}{GMm}(-\frac{GM}{r^3}\vec{r}) \times (mr^2\vec{\omega}) - \vec{\omega} \times \hat{r} = -\hat{r} \times \vec{\omega} - \vec{\omega} \times \hat{r} = 0$$

### 2.2 The setup

Because the angular momentum does not change, if we take $z$ to be the axis perpendicular to the plane of motion we can write $\frac{\vec{v} \times \vec{L}}{GMm} = \frac{L}{GMm}\vec{v} \times \hat{z}$, but actually $k \equiv \frac{L}{GMm}$ is constant and $\vec{v} \times \hat{z}$ is just a rotated version of $\vec{v}$ by 90 degrees (the rotation will be in the same direction for any of the speeds at points

$A$, $B$ and $C$). Now, take a point $O$ in the plane and draw 3 vectors from it, one to the left $OA$, the other up $OB$, and the last one to the right $OC$, so that the first and second are orthogonal and the first and third are exactly opposite. Regarding the magnitudes, choose the first as $kv_1$, the second $kv_2$, and the third $kv_3$. That is the same as depicting $\frac{L}{GMm}\vec{v_1} \times \hat{z}$, $\frac{L}{GMm}\vec{v_2} \times \hat{z}$ and $\frac{L}{GMm}\vec{v_3} \times \hat{z}$, except the plane is rotated 90 degrees. But because the eccentricity vector is constant, we have that subtracting the unit vector $\hat{r}$ should get us to the same point $E$ (giving us $e = |\vec{\varepsilon}| = OE$, the eccentricity we are looking for), but $\hat{r}$ has a constant magnitude, meaning that the point is an equal distance away from any of the points $A$, $B$ or $C$ - but this means that $E$ is the circumcenter of the triangle $ABC$!

## 3 Geometry




Since $E$ is the circumcenter, it is the intersection of perpendicular bisectors of the sides of the triangle. Knowing that $EA = EB = EC = 1$, we can find $OE$ (because we know the ratios of the triangle sides). To make everything much simpler, I will introduce a few substitutions:

$$\boxed{a = \sqrt{v_2^2 + v_3^2}}$$

$$\boxed{b = v_1 + v_3}$$

$$\boxed{c = \sqrt{v_1^2 + v_2^2}}$$

$$\boxed{p = \frac{a+b+c}{2}}$$

Let us start by expressing the eccentricity using the known velocities and the constant $k$. Let $D$ be the midpoint of $AC$. Then $(OD)^2 = e_x^2 = (k\frac{v_1+v_3}{2} -$

$kv_1)^2 = k^2(\frac{v_3-v_1}{2})^2$, and $(DE)^2 = e_y^2 = 1 - k^2(\frac{v_1+v_3}{2})^2$. So:

$$e = \sqrt{e_x^2 + e_y^2} = \sqrt{1 - k^2 v_1 v_3}$$

For the constant $k$, it can be found by exploiting the fact that $EA = EB = EC = 1$ and two ways to find the area of a triangle:

$$S = \sqrt{kp(kp-ka)(kp-kb)(kp-kc)} = \frac{kakbkc}{4 \cdot AE} = k^3 \frac{abc}{4}$$

$$k = \frac{4\sqrt{p(p-a)(p-b)(p-c)}}{abc}$$

$$k^2 = \frac{16p(p-a)(p-b)(p-c)}{a^2b^2c^2}$$

Substituting this into the expression for $e$ gives:

$$\boxed{e = \sqrt{1 - \frac{16p(p-a)(p-b)(p-c)v_1v_3}{a^2b^2c^2}}}$$

where

$$\boxed{a = \sqrt{v_2^2 + v_3^2}}$$

$$\boxed{b = v_1 + v_3}$$

$$\boxed{c = \sqrt{v_1^2 + v_2^2}}$$

$$\boxed{p = \frac{a+b+c}{2}}$$

For the given case of velocities, substituting the values gives $e = \sqrt{\frac{17}{65}} \approx 0.511408312$.