Problem 5

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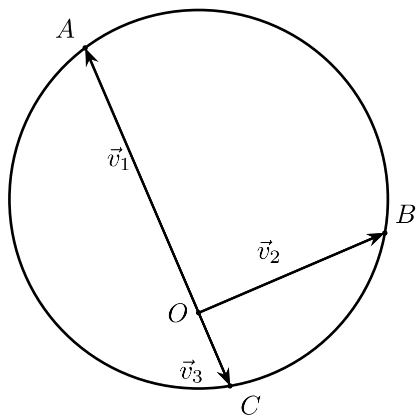
The 4th of March 2024

1 Condition

A satellite orbits a planet. At point A, its speed is v_1. At point B, its speed is v_2 and its velocity vector forms a right angle with the velocity vector at point A At point C, the velocity is exactly opposite to the velocity at point A, with a magnitude of v_3. Find the eccentricity of the orbit. Also determine the exact numerical value of the eccentricity when $v_1 = 1$ km/s, $v_2 = 2$ km/s и $v_3 = 3$ km/s.

2 Solution

First, let's draw a velocity hodograph and plot the velocities from the condition. Let's also recall the well-known fact that it is a circle the proof of this can be found at the end of the solution.) and that all velocities come from a point shifted relative to the center.



Define the radius of the circle. To do this, calculate the lengths of the sides and the area of the triangle ABC.

$$AB = \sqrt{v_1^2 + v_2^2}; \quad BC = \sqrt{v_2^2 + v_3^2}; \quad AB = v_1 + v_3.$$

$$S_{ABC} = \frac{AC \cdot OB}{2} = \frac{v_2(v_1 + v_3)}{2}$$

To do this, consider the triangle ABC and recall the following formula for the radius.

$$R = \frac{AB \cdot BC \cdot CA}{4S_{ABC}} = \frac{\sqrt{(v_1^2 + v_2^2) \cdot (v_2^2 + v_3^2)}}{2v_2}$$

Now let's recall one interesting fact about the intersection of chords. If the XY chord intersects with the ZW chord at point O, then the following equality is true: $XO \cdot YO = ZO \cdot WO$. Let's take advantage of this fact and equate $v_1 \cdot v_3 = v_a \cdot v_p$, where v_a this is the apocentric velocity, and v_p is the pericentric velocity.

$$v_a \cdot v_p = \sqrt{\frac{GM}{a}\frac{1-e}{1+e}} \cdot \sqrt{\frac{GM}{a}\frac{1+e}{1-e}} = \frac{GM}{a}, \quad \Longrightarrow \quad \sqrt{v_1 \cdot v_3} = \sqrt{\frac{GM}{a}}$$

Recall that the radius of the circle of the hodograph is expressed by the following formula (where p is the focal parameter of the orbit):

$$R = \sqrt{\frac{GM}{p}} = \sqrt{\frac{GM}{a} \cdot \frac{1}{1-e^2}}, \quad \Longrightarrow \quad \frac{R}{\sqrt{v_1 \cdot v_3}} = \sqrt{\frac{1}{1-e^2}}, \quad \Longrightarrow \quad \frac{v_1 \cdot v_3}{R^2} = 1 - e^2, \quad \Longrightarrow \quad e = \sqrt{1 - \frac{v_1 \cdot v_3}{R^2}}$$

$$e = \sqrt{1 - \frac{4v_2^2 \cdot v_1 \cdot v_3}{(v_1^2 + v_2^2) \cdot (v_2^2 + v_3^2)}}$$

Substituting the numerical values from the problem, we get:

$$e = \sqrt{1 - \frac{48}{65}} = 0.5114083$$

3 Proof of the hodograph

To begin with, it's worth talking about such a thing as a velocity hodograph. This is the geometric location of the ends of the velocity vectors in our case of the satellite. To begin with, we will prove that for circular and elliptical orbits it will be a circle.

First, let's prove Kepler's second law. If the interaction force is central, then the angular momentum relative to the center of force remains constant.

$$\frac{d\vec{L}}{dt} = \vec{M}, \quad \Longrightarrow \quad d\vec{L} = \vec{0}.$$

The angular momentum module

$$L = mv_{\perp}r = \text{const}.$$

Then in a short time dt the body «sweeps» the area

$$\frac{1}{2} \cdot rv_{\perp} \cdot dt = dS.$$

Note thatr $r \cdot v_{\perp} = \text{const}$, then in equal time intervals, the body sweeps an equal area.

Let's divide the ellipse trajectory into small sections with the same angular magnitude $d\alpha$ and calculate the change in the velocity vector in these sections. Let's write down Newton's second law

$$\vec{F} = m\frac{d\vec{v}}{dt}.$$

Here dt — the time it takes for a body to pass a sector, $d\vec{v}$ — speed vector change over time dt.The force is determined by the law of universal gravitation

$$\vec{F} = \frac{\vec{A}}{R^2},$$

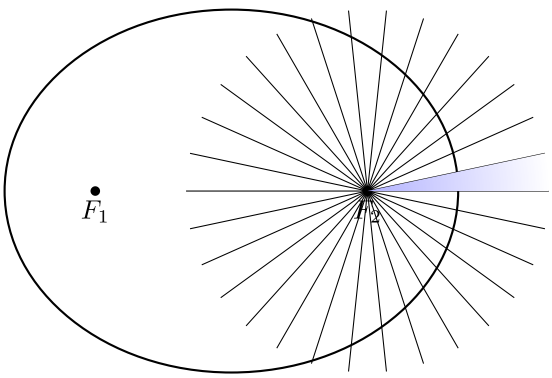
Where R is the distance from the center of force to the body, and $\vec{\alpha}$ is a constant coefficient in magnitude, depending on the masses of the body and the planet, directed toward the center of force. The time dt is determined by the area swept out:

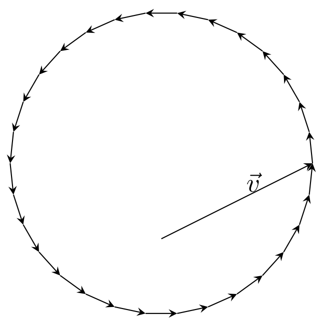
$$dt \propto dS \propto R^2.$$

Then,

$$|d\vec{v}| = \frac{1}{m}\left|\vec{F}\right| \cdot dt \propto \frac{1}{m}\frac{1}{R^2} \cdot R^2 = \text{const}, \quad \Longrightarrow \quad |d\vec{v}| = \text{const}.$$

We have shown that when the radius vector rotates through equal small angles $d\alpha$, the velocity of the body changes by the same amount. Note that the direction of $d\vec{v}$ also rotates by the angle $d\alpha$. Therefore, the velocity hodograph is a circle (see the figure).





We will also calculate the radius. This can be done by taking the arithmetic mean of the geocentric velocity and the recentric velocity:

$$R = \frac{v_p + v_a}{2} = \sqrt{\frac{GM}{4a}} \cdot \left(\sqrt{\frac{1+e}{1-e}} + \sqrt{\frac{1-e}{1+e}}\right) = \sqrt{\frac{GM}{4a}} \cdot \frac{2}{\sqrt{1-e^2}} = \sqrt{\frac{GM}{p}}$$