Physicscup 2025 - Problem 5

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1 Introduction

This solution uses the fact that the hodograph for the satellite is a a circle as the gravitation force has a radial symmetry and follows an inverse-square-law with respect to do the distance. Before we use this fact we want to proof it. Afterwards we will show how this hodograph leads to a short solution.

2 Lemmas

For our purpose we need three intermediate results. First, we need to show eccentricity vector is conserved. It is proportional to the LRL-vector (Laplace-Runge-Lenz). Second, that this vector indeed has the eccentricity as an absolute value. And third, from the eccentricity vector we can derive the shape of the hodograph.
All of these proofs are already well-known so there is no need to reinvent them. Instead, I want to point out that both Wikipedia (*Laplace–Runge–Lenz-vector*) and last years Physicscup (*PC2024-P4 Best solutions and final results*) show nice proofs, for the LRL-vector there are even numerous ways. Still, we for completeness we want to proof all necessary intermediate results here.

2.1 Lemma 1: Conservation of eccentricity vector

The eccentricity vector is defined as $\vec{\varepsilon} = \frac{\vec{A}}{mk} = \frac{\vec{p}\times\vec{L}}{mk} - \hat{r}$. Its time derivative vanishes:

$$\dot{\vec{\varepsilon}} = \frac{d}{dt}\left[\frac{\vec{p}\times\vec{L}}{mk} - \hat{r}\right] = \frac{m\frac{d\vec{v}}{dt}\times(\vec{r}\times m\vec{v})}{mk} - \frac{d\hat{r}}{dt} = \frac{\left(\frac{-k\hat{r}}{r^2}\right)\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)}{k} - \frac{d\hat{r}}{dt} = \frac{-k}{r^2}\frac{\vec{r}\left(\hat{r}\cdot\frac{d\vec{r}}{dt}\right) - r\frac{d\vec{r}}{dt}}{k} - \frac{d\hat{r}}{dt}$$
$$= -\frac{\hat{r}}{r}\left(\frac{1}{2r}\frac{d(r^2)}{dt}\right) + \frac{1}{r}\frac{dr}{dt}\hat{r} + \frac{r^2}{r^2}\frac{d\hat{r}}{dt} - \frac{d\hat{r}}{dt} = -\frac{\hat{r}}{r}\frac{dr}{dt} + \hat{r}\frac{1}{r}\frac{dr}{dt} = 0$$

with $k = GMm$ and the angular momentum $\vec{L}$ being a conserved quantity.

2.2 Lemma 2: The absolute value of $|\vec{\varepsilon}|$

$$|\varepsilon|^2 = \vec{\varepsilon}\cdot\vec{\varepsilon} = \left(\frac{\vec{p}\times\vec{L}}{mk} - \hat{r}\right)^2 = \frac{\left|\vec{p}\times\vec{L}\right|^2}{(mk)^2} + 1 - 2\frac{\vec{p}\times\vec{L}}{mk}\cdot\hat{r} = \frac{(pL)^2}{(mk)^2} + 1 - 2\frac{L^2}{mkr} = 1 + \frac{L^2}{m^2k^2}\left(m^2v^2 - 2m\frac{k}{r}\right)$$
$$= 1 + \frac{L^2}{2m^2k^2}\left(\frac{m}{2}v^2 - \frac{k}{r}\right) = 1 + \frac{2EL^2}{m^2k^2}$$

which is indeed the square of the eccentricity. Here, we used the scalar triple product $\vec{r}\cdot\left(\vec{p}\times\vec{L}\right) = (\vec{r}\times\vec{p})\cdot\vec{L} = \vec{L}\cdot\vec{L} = L^2$ and the fact that the angular moment is perpendicular to the plane of motion.

2.3 Lemma 3: Hodograph is a circle

We already know that the LRL-vector is conserved. We can rewrite it and the take the dot product of itself:

$$mk\hat{r} = \vec{p}\times\vec{L} - \vec{A} \Rightarrow m^2k^2 = \left(\vec{p}\times\vec{L} - \vec{A}\right)^2 = \left|\vec{p}\times\vec{L}\right|^2 + A^2 - 2(\vec{p}\times\vec{L})\cdot\vec{A} = p^2L^2 + A^2 - 2(\vec{A}\times\vec{p})\cdot\vec{L}$$

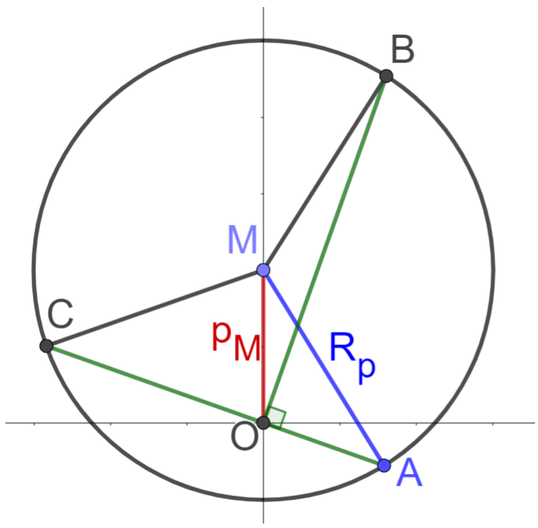
by again using the scalar triple product rule. Without loss of generality (w.l.o.g.) we can choose $\vec{L}$ along the z-axis and the major semiaxis along the x-axis:

$$m^2k^2 = p^2L^2 + A^2 - 2(Ap_y\hat{z})\cdot L\hat{z} = p^2L^2 + A^2 - 2Ap_yL = p_x^2L^2 + (p_yL)^2 - 2(p_yL)A + A^2 = p_x^2L^2 + (p_yL - A)^2$$
$$\Leftrightarrow p_x^2 + \left(p_y - \frac{A}{L}\right)^2 = \left(\frac{mk}{L}\right)^2$$

This equation is called the locus equation.

3 Solution

Now we want to efficiently use these facts. First, we observe that the hodograph is a circle with radius $\frac{mk}{L} =: R_p$ and the center of the circle has coordinates $(p_{xM}, p_{yM}) = (0, \frac{A}{L})$ so the length of the vector is $p_M = p_{yM} = \frac{A}{L}$. So the eccentricity is simply $\varepsilon = |\vec{\varepsilon}| = \left|\frac{\vec{A}}{mk}\right| = \frac{A}{mk} = \frac{p_M}{R_p}$.
Next from the text the three velocities v_1, v_2, v_3 are given and in addition we know that $\vec{v_1}\perp\vec{v_2}\perp\vec{v_3}$. It is not a problem that only the velocities and not the momenta are given as we are not even interested in the values of p_M and R_p but only the ratio and so the mass cancels out. Also, as all the velocities are perpedicular to the angular momentum (also conserved) they all lie in the plane of motion. We want to use the geometric connections between the momenta p_1, p_2, p_3 to relate them to p_M and R_p.



Lets draw a sketch of the hodograph. For the sketch, we choose some arbitrary point A on the circle and then constructed B and C accordingly. The center of the coordinate system is denoted by O and the center of the circle is M. Now we want to relate $|\overline{CO}| = p_3$ and $|\overline{BO}| = p_2$ to $|\overline{AO}| = p_1$ and we do this via the law of cosines.
Because point C and A must be opposite in the hodograph O lies on the line $\overline{AC}$ and therefore $\angle MAO = \angle MAC$ holds. We apply the law of cosines to the triangles $\triangle AOM$

and $\triangle ACM$:

$$|\overline{MO}|^2 = p_M^2 = |\overline{MA}|^2 + |\overline{AO}|^2 - 2\cdot|\overline{MA}|\cdot|\overline{AO}|\cos(\angle MAO) = R_p^2 + p_1^2 - 2R_p p_1 \cos(\angle MAO)$$

$$|\overline{CM}|^2 = R_p^2 = |\overline{MA}|^2 + |\overline{CA}|^2 - 2\cdot|\overline{MA}|\cdot|\overline{CA}|\cos(\angle MAC) = R_p^2 + (p_1+p_3)^2 - 2R_p(p_1+p_3)\frac{R_p^2+p_1^2-p_M^2}{2R_p p_1}$$

$$\Leftrightarrow (p_1+p_3)^2 = (p_1+p_3)\frac{R_p^2+p_1^2-p_M^2}{p_1}$$

$$\Leftrightarrow p_1(p_1+p_3) = p_1^2 + p_1p_3 = R_p^2 + p_1^2 - p_M^2$$

$$\Leftrightarrow p_1p_3 = R_p^2 - p_M^2$$

Next, we use a similar approach to connect p_1 to p_2 here using the law of cosines for the triangles $\triangle OMB$ and $\triangle AOM$. To figure out the angle $\angle BOM$ we use the fact that $\angle BOM = \angle AOM - \angle AOB = \angle AOM - 90°$ holds:

$$|\overline{MB}|^2 = R_p^2 = |\overline{MO}|^2 + |\overline{OB}|^2 - 2\cdot|\overline{MO}|\cdot|\overline{OB}|\cos(\angle MAO) = p_M^2 + p_2^2 - 2p_M p_2\cos(\angle BOM)$$

$$\Leftrightarrow 2p_M p_2 \cos(\angle BOM) = p_2^2 + (p_M^2 - R_p^2) = p_2^2 - p_1p_3$$

$$\cos(\angle BOM) = \cos(\angle AOM - 90°) = \cos(\angle AOM)\cos(90°) + \sin(\angle AOM)\sin(90°) = \sin(\angle AOM)$$

$$\Rightarrow (p_2^2 - p_1p_3)^2 = (2p_M p_2\cos(\angle BOM))^2 = (2p_M p_2 \sin(\angle AOM))^2 = 4p_M^2p_2^2\left(1-\cos(\angle AOM)^2\right)$$

$$= 4p_M^2p_2^2\frac{4p_M^2p_1^2 - (p_1^2+p_M^2-R_p^2)^2}{4p_M^2p_1^2} = \frac{p_2^2}{p_1^2}\left(4p_M^2p_1^2 - (p_1^2-p_1p_3)^2\right) = p_2^2\left(4p_M^2 - (p_1-p_3)^2\right)$$

$$\Leftrightarrow p_M^2 = \frac{1}{4}\left(\frac{(p_2^2-p_1p_3)^2}{p_2^2} + (p_1-p_3)^2\right)$$

At this point we can calculate p_M and thereby also R_p which is sufficient to finally obtain the result for the eccentricity:

$$\varepsilon = \sqrt{\frac{p_M^2}{R_p^2}} = \sqrt{\frac{p_M^2}{p_M^2+p_1p_3}} = \sqrt{1-\frac{p_1p_3}{\frac{(p_2^2-p_1p_3)^2}{4p_2^2}+\frac{(p_1-p_3)^2}{4}+p_1p_3}} = \sqrt{1-\frac{p_1p_3}{\frac{(p_2^2-p_1p_3)^2}{4p_2^2}+\frac{(p_1+p_3)^2}{4}}}$$

$$= \sqrt{1-\frac{v_1v_3}{\frac{(v_2^2-v_1v_3)^2}{4v_2^2}+\frac{(v_1+v_3)^2}{4}}} = \sqrt{1-\frac{1\cdot 3}{\frac{(2^2-1\cdot3)^2}{4\cdot2^2}+\frac{(1+3)^2}{4}}} = \sqrt{1-\frac{3}{\frac{1^2}{16}+\frac{4^2}{4}}} = \sqrt{1-\frac{3\cdot16}{1+4\cdot16}} = \sqrt{1-\frac{48}{65}}$$

$$= \sqrt{\frac{17}{65}} \approx 0.5114$$

In all the calculations we used the assumption that the orbit is an ellipse so the satellite is bounded to its planet. Without this assumption the point would lie outside of the hodograph. Fortunately, this assumption must be true. It is stated in the text that "the velocity is exactly opposite to the velocity at point A" which means that the line $\overline{AC}$ must contain point O which is only possible for the case of an ellipse ($\varepsilon < 1$) which also explains why the root is always well defined.