2025 Physics Cup Problem 5 Solution

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§1 Solution

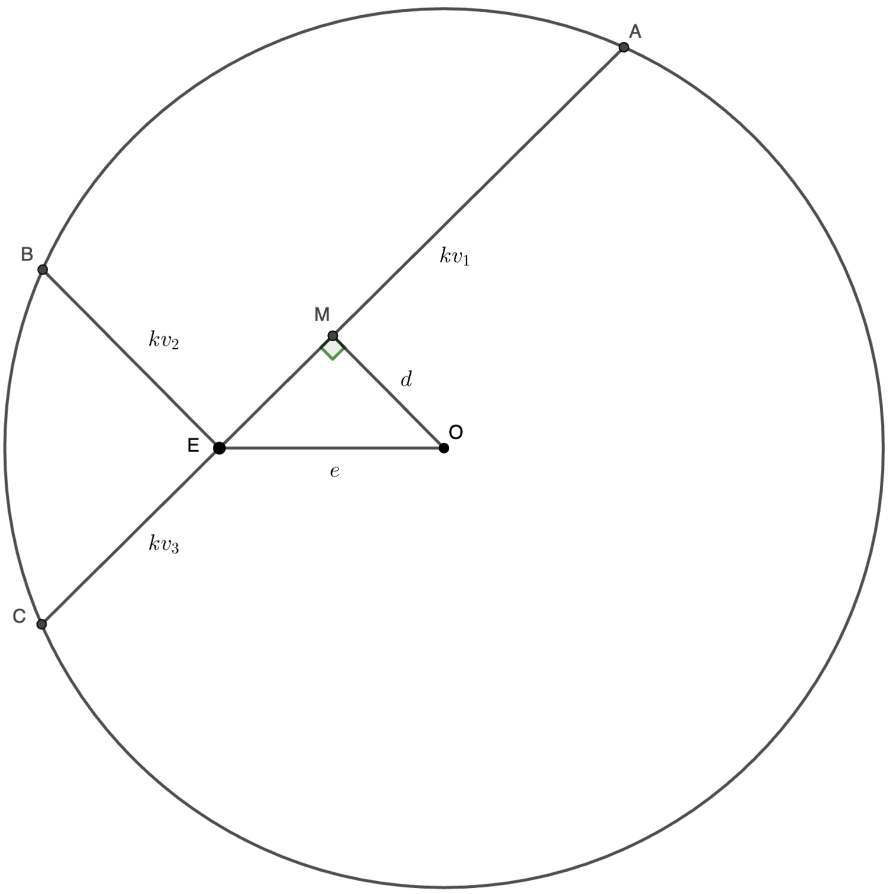
We will make use of the conserved eccentricity vector[1]

$$\mathbf{e} = \frac{\mathbf{p} \times \mathbf{L}}{GMm^2} - \hat{\mathbf{r}} \implies \mathbf{v} \times \mathbf{L} = GMm(\mathbf{e} + \hat{\mathbf{r}})$$

Note that $\mathbf{L}$ is always perpendicular to the orbit plane so $|\mathbf{v} \times \mathbf{L}| = |\mathbf{v}||\mathbf{L}|$, so

$$k := \frac{|\mathbf{e} + \hat{\mathbf{r}}_\mathbf{A}|}{v_1} = \frac{|\mathbf{e} + \hat{\mathbf{r}}_\mathbf{B}|}{v_2} = \frac{|\mathbf{e} + \hat{\mathbf{r}}_\mathbf{C}|}{v_3} \tag{1}$$

Also, for the pair of positions A and B, since $\mathbf{v_A} \perp \mathbf{v_B}$, we have $\mathbf{e} + \hat{\mathbf{r}}_\mathbf{A} \perp \mathbf{e} + \hat{\mathbf{r}}_\mathbf{B}$. Similarly, for positions A and C, we have that $\mathbf{e} + \hat{\mathbf{r}}_\mathbf{A}$ points opposite to $\mathbf{e} + \hat{\mathbf{r}}_\mathbf{C}$ which also implies that $\mathbf{e} + \hat{\mathbf{r}}_\mathbf{C} \perp \mathbf{e} + \hat{\mathbf{r}}_\mathbf{B}$.



[1] A proof of the conservation of $\mathbf{e}$ will be in the remarks.

Now, we graph these vectors (refer to diagram on last page). Let $E = (0, 0)$ and $\overrightarrow{EO} = \mathbf{e}$. Point A represents $\mathbf{e} + \hat{\mathbf{r}}_\mathbf{A}$ so $OA = |\hat{\mathbf{r}}_\mathbf{A}| = 1$. Since similar holds for B and C, we have A,B, and C lying on the circle of radius 1 centered at O. Lastly, from equation 1, we have $EA = kv_1$, $EB = kv_2$, and $EC = kv_3$. To finish, we just use some Euclidean geometry.

Power of a point on E tells us

$$k^2 v_1 v_3 = kv_2(kv_2 + 2d) = 1 - e^2$$

Pythagorean theorem says

$$d^2 = e^2 - \frac{k^2(v_1 - v_3)^2}{4}$$

The first equation yields $2v_2 d = k(v_1 v_3 - v_2^2)$ so

$$\begin{aligned} k^2(v_2^2 - v_1 v_3)^2 &= 4v_2^2 d^2 \\ &= 4v_2^2 \left(e^2 - \frac{k^2(v_1 - v_3)^2}{4}\right) \\ &= 4v_2^2(1 - k^2 v_1 v_3) - v_2^2 k^2 (v_1 - v_3)^2 \\ &= 4v_2^2 - 2k^2 v_1 v_2^2 v_3 - k^2 v_1^2 v_2^2 - k^2 v_2^2 v_3^2 \end{aligned}$$

$$\implies k^2 v_2^4 + k^2 v_1^2 v_3^2 = 4v_2^2 - k^2 v_1^2 v_2^2 - k^2 v_2^2 v_3^2$$

$$\implies k^2 = \frac{4v_2^2}{v_2^4 + v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2}$$

Since $e = \sqrt{1 - k^2 v_1 v_3}$,

$$\boxed{e = \sqrt{1 - \frac{4 v_1 v_2^2 v_3}{v_2^4 + v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2}}} = \boxed{\sqrt{\frac{17}{65}}}$$

for the given values.

Remark. Conservation of $\mathbf{e}$ holds since

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{1}{GMm^2}(\mathbf{F} \times \mathbf{L}) - \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = -\dot{\theta}\,(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) - (\dot{\theta}\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = 0$$