2025 Physics Cup Problem 4

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Preliminaries

We use Γ to denote the circle and denote its elliptical image by $\mathcal{E}$. Let center of the lens be O and the focus of the lens be F. Let the focal length be f. In this solution, we will assume without loss of generality that O coincides with the origin and F is located at $(f, 0)$. These coordinates do not necessarily correspond to the same coordinates in the given configuration.

Lemma 1. *The image of a line is a line.*

Proof. A light ray coinciding with the line must pass through the images of each point on the line, and thus the images must all lie on the the refracted light ray. □

This proof provides physical intuition, but is problematic if the original line is parallel to the lens. A more mathematically rigorous proof follows from the fact that the image of a point (x, y) is given by

$$(x, y) \mapsto \left(\frac{f}{f+x} x, \frac{f}{f+x} y \right),$$

which is a direct consequence of the lens equation. In homogeneous coordinates, this corresponds to the transformation

$$(x : y : z) \mapsto (fx : fy : fz + x),$$

which is a linear transformation (homography) in the projective plane that maps lines to lines. As a corollary, the image of a tangent line to Γ is a line tangent to $\mathcal{E}$, since the bijectivity of the projective transformation ensures that the image intersects $\mathcal{E}$ exactly once. Let the center of Γ be M, and let the intersection points of OM with Γ by C and D. Let C' and D' by the images of C and D, respectively.

Lemma 2. *The following cross-ratio relation holds:*

$$(C, D; M, O) = (C', D'; M'O')$$

Proof. Let C'' be the projection of C' onto the vertical axis of the lens and define M'', D'' similarly. We have

$$(C, D; M, O) \overset{\infty}{\cong} (C'', D''; M'', O) \overset{F}{=} (C', D'; M', O).$$

□

Again, this result can be obtained directly by citing the fact that a projective transformation preserves cross ratios. Let the tangents to $\mathcal{E}$ at C' and D' intersect at H.

Lemma 3. *H lies on focal plane of the lens.*

Proof. Note that the tangents at C' and D' are the images of the tangents to Γ at C and D, so H is the image of the intersection of those tangents. Since the tangents to Γ are parallel, their intersection is at infinity, and the distance of the image to the lens is

$$\lim_{x \to \infty} \frac{fx}{f+x} = f,$$

which lies on the focal plane by definition. As a corollary, $OF \perp FH$, so F must lie on a circle with diameter OH. □

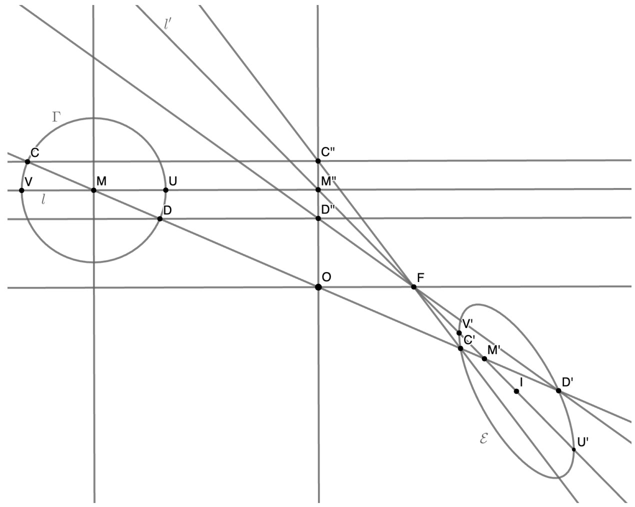

Figure 1: Diagram for Lemmas 2 and 4

Let I denote the center of $\mathcal{E}$.

Lemma 4. *The points I, M', and F are collinear.*

Proof. Consider the line l parallel to the optical axis passing through M. The image of l will be a line passing through F and M'. Note that l passes through the leftmost and rightmost points U, V of Γ. Excluding the possibility that Γ passes through $x = -f$, in which case its image would be a hyperbola, the distance of the image of a point to the lens is a monotonic function of the distance of the point to the lens. Hence, l must pass through the images of U and V, which are the leftmost and rightmost points U' and V' of $\mathcal{E}$. Since I lies on $U'V'$, it must lie on l. □

Solution

Using the lemmas above, we know that the focus of the lens must lie on the intersection of IM' and the circle with diameter OH. H is straightforward to construct, so we only need to find M' in order to determine F, and thus the optical axis OF. Suppose the tangents from O to Γ are tangent at A and B. Since these lines map to themselves, they coincide with the tangents from O to $\mathcal{E}$, which are tangent at the images A', B' of A, B.

Note that any circle Γ^* tangent to OA and OB will be homothetic to Γ with center O. Hence, if Γ^* has center M^* on OM and OM intersects Γ^* at C^* and D^*, then

$$(C^*, D^*; M^*, O) = (C, D; M, O).$$

Hence, using Lemma 2, we can chase around this cross ratio using perspectivites in order to construct M'. The way this is done in the construction is by first projecting the points onto OH and then back to OM, so that the cross ratio is preserved.

Steps

The full construction is given as follows:

1. Construct tangents from O onto $\mathcal{E}$ and mark the tangency points A' and B'.
2. Construct the angle bisector ℓ of $\angle A'OB'$, mark intersections with $\mathcal{E}$ as C' and D'.
3. Construct the center of $\mathcal{E}$ and mark it I.
4. Construct tangents to $\mathcal{E}$ at C' and D' and mark their intersection point H.
5. Construct the circle Ω with diameter OH. (In Geogebra, we can first mark the midpoint of OH).
6. Pick an arbitrary point M^* on ℓ and draw a circle ω^* centered at M^* and tangent to OA' and OB'. (In Geogebra, we can first mark the projection of $M*$ onto OA').
7. Mark the intersection points of ω^* with ℓ as C^* and D^* (see Appendix).
8. Pick an arbitrary point P. Mark the intersections of PC^*, PM^*, and PD^* with OH as C^{**}, M^{**}, and D^{**}, respectively.
9. Draw $D^{**}D'$ and $C^{**}C'$ and mark their intersection point Q.
10. Draw line $M^{**}Q$ and mark its intersection with ℓ as M'.
11. Draw line IM' and mark its intersection with Ω as F (see Appendix).
12. Draw line OF, which is the optical axis of the lens.

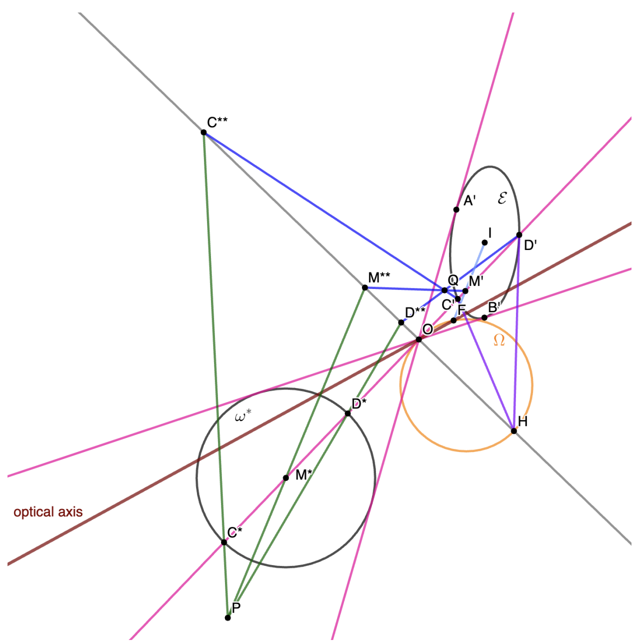

Figure 2: Diagram of full construction

Answer

In the given diagram, the slope of the optical axis is $a = 0.54627393$.

Appendix: Ambiguities in construction

Note that there is ambiguity in choosing C^* and D^*, as well as choosing which intersection point is F. In general, there will be four possible foci to choose from. However, careful inspection of each of the possibilities rules out all but one.

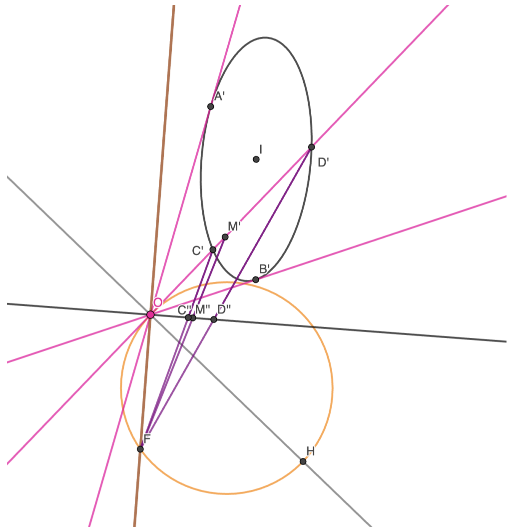

Figure 3: Alternate configuration I: The other intersection of IM' is chosen to be F. In this case, the projection M'' is not the midpoint of C'' and D'', which does not allow us to construct Γ.

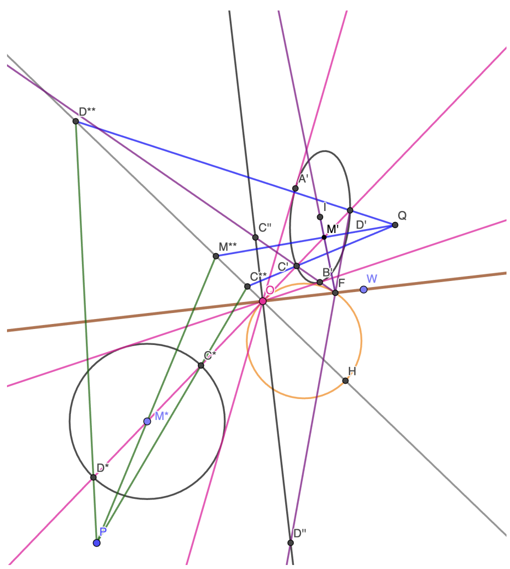

Figure 4: Alternate configuration II: The choice for C^* and D^* has been switched. In this case, the projection M'' is not even between C'' and D'', which is incorrect.

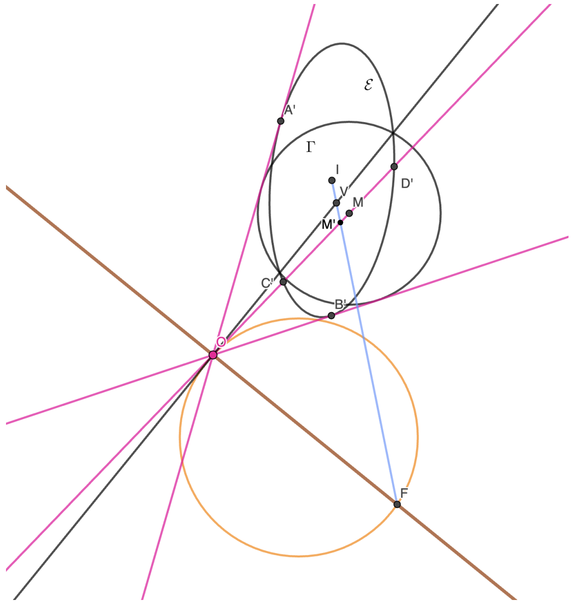

Figure 5: Alternate configuration III: Same as alternate configuration II, except the other intersection point of IM' is chosen. In this case, we are actually able to construct Γ and we get that the ellipse is the correct image of Γ under geometric construction. However, this configuration poses physical difficulties due to the intersection of the vertical axis of the lens with Γ, which prevents light one side from illuminating the entire circle. Hence, this configuration will be excluded.

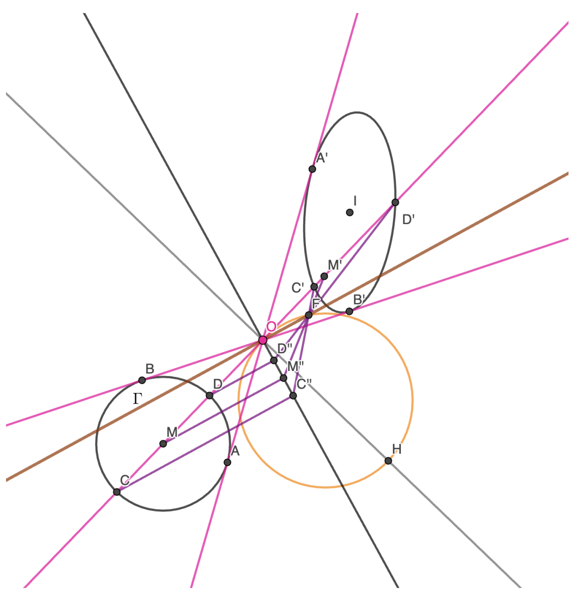

Figure 6: This is the correct choice for F. The image of Γ is exactly the ellipse.