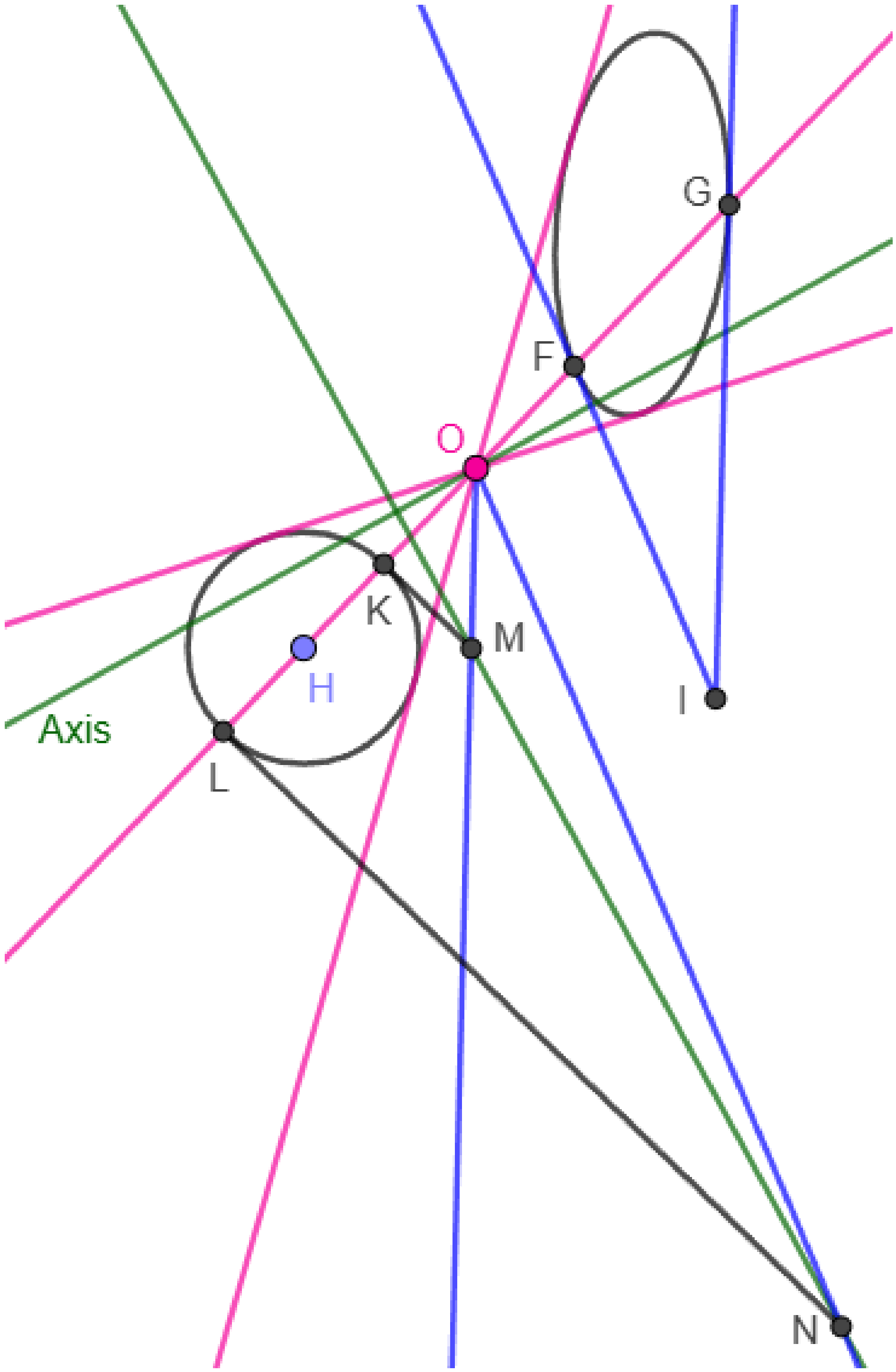
Problem 4



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First, we note that a point and its image are collinear with O, so a line through O is its own image. Thus, the two tangents to the ellipse through O are also tangent to the image circle. Construct the bisector of these tangents, which contains the center of the image circle.

Let the bisector intersect the ellipse at F and G, and let I be the intersection of the tangents to the ellipse at F and G. For the moment, assume that we know where the image circle is located; let the bisector intersect the circle at L and K. Let M be the intersection of the tangent to the circle at K and the focal line on the circle's side, and define N similarly.

Note that lines KM and IG are images, and LN and IF are images. Because M lies on the focal line, it maps to the point at infinity on ray IG, so $OM \| IG$. Similarly, $ON \| IF$.

Now, we can work backwards. Construct an arbitrary circle tangent to the ellipse's tangents. Define K and L be the intersections of the circle with the bisector. Define M to be the intersection of the tangent at K with the parallel to IG through O, and define N similarly. Then, we know that MN is parallel to the focal line through a homothety centered at O, letting us construct the lens axis as the perpendicular to MN through O.