Physics Cup 2025 Problem 4 Solution (Second Attempt)

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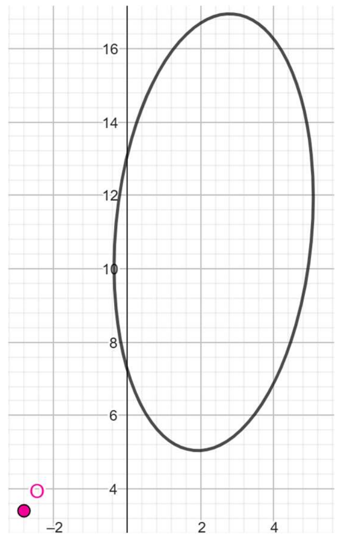
Problem: In the figure below, the ellipse is a real image of a circle created by an ideal thin lens whose center is at point *O*. Geometrically construct the main optical axis of this thin lens, which is known to lie in the plane of this figure.



Answer: The main optical axis satisfies the line -10.6843541831x + 19.5586016308y = 96.5613179146. If we express it in term of y = ax+b, then

$$a = 0.5462739303$$
$$b = 4.9370256492$$

Solution: We require two information.

1. The absolute difference of the reciprocal length of two points on a line passing through the center of the lens, is the same as the difference of the reciprocal length of their images.
2. Inversion of a point with respect to unit circle centered at origin gives the reciprocal length.

Information 1: The absolute difference of the reciprocal length of two points on a line passing through the center of the lens remains the same as their images.

Consider the lens equation, $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$. and consider one point on a line passing through the center. Suppose the line makes angle α with the principal axis. Multiply the lens equation by cos α gives

$$\frac{\cos\alpha}{u} + \frac{\cos\alpha}{v} = \frac{\cos\alpha}{f} \Rightarrow \frac{1}{l_u} + \frac{1}{l_v} = \frac{\cos\alpha}{f}$$

Where we denote l_u as the length of the point on the line from the center.

Now, we do the same for a second point on the line and we take the difference, we have

$$\frac{1}{l_{u,1}} - \frac{1}{l_{u,2}} = \frac{1}{l_{v,2}} - \frac{1}{l_{v,1}}$$

Hence, the absolute difference remains unchanged. Now, if we take the angle bisector of the tangents from the center to the ellipse, the angle bisector cuts the ellipse at the points which are the image of diammetrical points on the circle. Interestingly, we can express the difference in reciprocal length of the diammetrical points in term of its radius and the angle bisector angle alone. We have

$$\frac{1}{l_{u,1}} - \frac{1}{l_{u,2}} = \frac{1}{l-R} - \frac{1}{l+R} = \frac{2R}{l^2 - R^2} = \frac{2\sin\theta}{l\,cos^2\theta} = \frac{2\sin\theta}{\frac{R}{\sin\theta}\,cos^2\theta} = \frac{2\sin\theta}{l'}$$

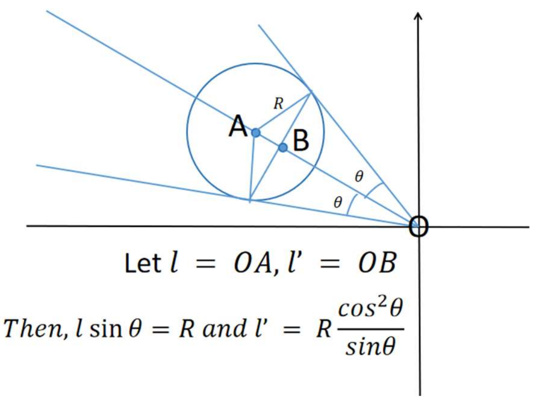


Figure 1. Schematic of circle and its tangents

Where l is the length from the center of the lens to center of circle and l' is the distance from the center of the lens to the chords connecting the tangent. 2θ is the angle between the two tangents, and R is the radius of the circle.

Hence, the strategies is to compute $\frac{1}{l_{u,1}} - \frac{1}{l_{u,2}}$ from the ellipse to calculate l' and after we have found point B, we can find the intersection between the line connecting the tangent and their respective image lines on the ellipse. Their intersection lies on the lens. Finally, the line perpendicular to the lens is the optical axis.

Information 2: Inversion of a point with respect to unit circle gives the reciprocal length.

Inversion of a point with respect to the circle satisfy

$$\mathrm{A} * \mathrm{A}' = \mathrm{r}^2$$

If $\mathrm{r} = 1$, then

$$\mathrm{A} = \frac{1}{\mathrm{A}'}$$

In the geogebra file, the blue line is the lens axis, while the green line denotes the optical axis.