Physics Cup 2025

Problem 4

Teo Kai Wen

Useful results

Here are some useful results of lenses that will come in handy:

1. The image of a line is also a line. Hence, lines passing through the center of a lens is its own image.

   This is easily proven by considering a line as a light ray. They are equivalent as a light ray is formed by constructive interference of a line of light sources, each a source of Huygens' wavelets. For the corollary, light rays passing through the center of a lens continue in the same line.

2. Homothety - Any point, its image and the center of the lens are collinear.

   This is also easily proven by considering a point as a light source, and its image is where the light rays converge. Light rays passing through the center of the lens continue in the same line so the image must lie on that line.

3. The images of parallel lines converge at the same point on the focal plane.

   It is well-known that parallel light rays are focused to the same point on the focal plane. In fact, that is the very definition of the focal plane. Treating lines as light rays, we get the desired result.

4. Along a line passing through the center of the lens, points further away from the lens have images closer to the lens.

Finding the image of the center of the circle

1. Motivated by homothety, we draw tangent lines to the ellipse through O, labelling the points as A' and B'.

2. We now construct the original circle on the other side of the lens (or a uniformly scaled version of it). By homothety, the circle must be tangent to OA' and OB'. First, bisect $\angle A'OB'$ and define the center of the circle at some arbitrary point M. We label the intersection points of the bisector with the ellipse as C' and D'.

   Our choice of M is arbitrary, as all circles tangent to the same two lines can be transformed to one another through scaling from O. This is explained in step 5.

3. A perpendicular is dropped from M to $A'O$ produced (not shown to keep the graph neat), and we label the intersection as A. A circle is made with M and A. We assign corresponding points, with their primed equivalents being the image of that point.

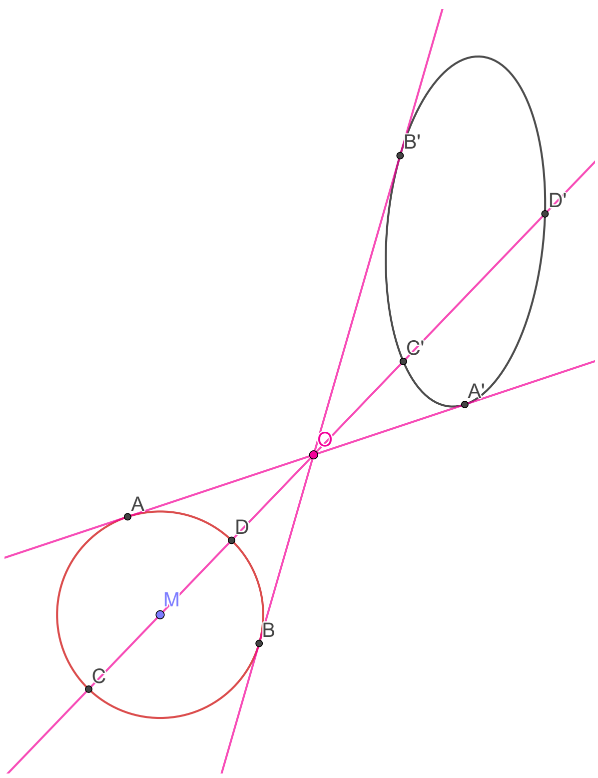

Figure 1: Steps 1-3

4. To find the image of the center of the circle, we will need to find another diameter. We draw a perpendicular line to OM through M, and label the intersection points with the circle E and F.

5. From homothety, we draw a line from E to O and from F to O. These lines produced will intersect the ellipse at the points E' and F'. Note that each line intersects the ellipse at 2 points, so we should pick the intersection point closer to O, as E and F are the further intersection points of the circle (see result 4). Notice that since all circles tangent to OA' and OB' can be transformed to one another from scaling through O, the choice of M can be arbitrary. You can test this by moving M along OM and noticing that the lines through O are unchanged.

6. Now that we have the images of two diameters, the intersection of these two images $E'F'$ and $C'D'$ is the image of the center of the circle M'.

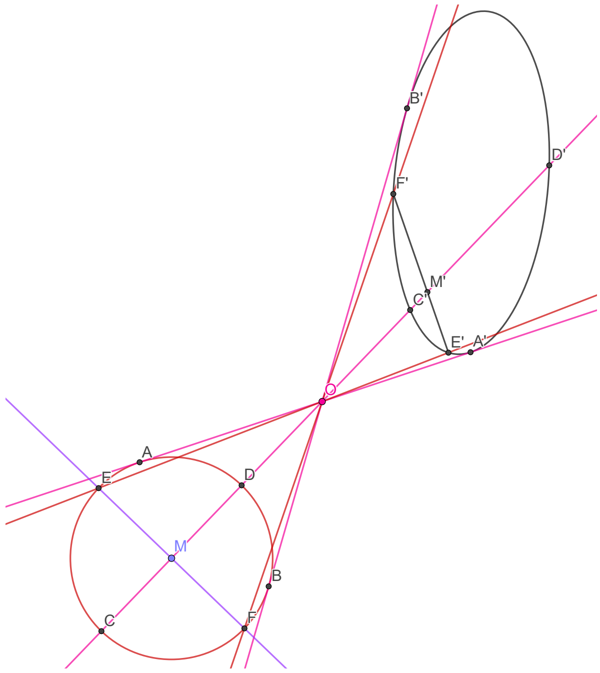

Figure 2: Steps 4-6

Finding the focal plane and the optical axis

7. Now that we have found M', previous construction lines have been removed for clarity. Motivated by result 1 and result 3, we construct lines parallel to AM and BM, passing through O. We label these l_A and l_B respectively (the purple lines). By result 1, lines passing through O are its own image. Applying result 3 as well, the intersection of l_A and $A'M'$ produced intersect on the focal plane, at F_2. Repeating with point B gives an intersection point F_1. We end up with two intersection points, F_1 and F_2 on the focal plane.
8. We wrap up by drawing a line between F_1 and F_2 and dropping a perpendicular through O to find the optical axis (bright orange line). See next page for final graph.

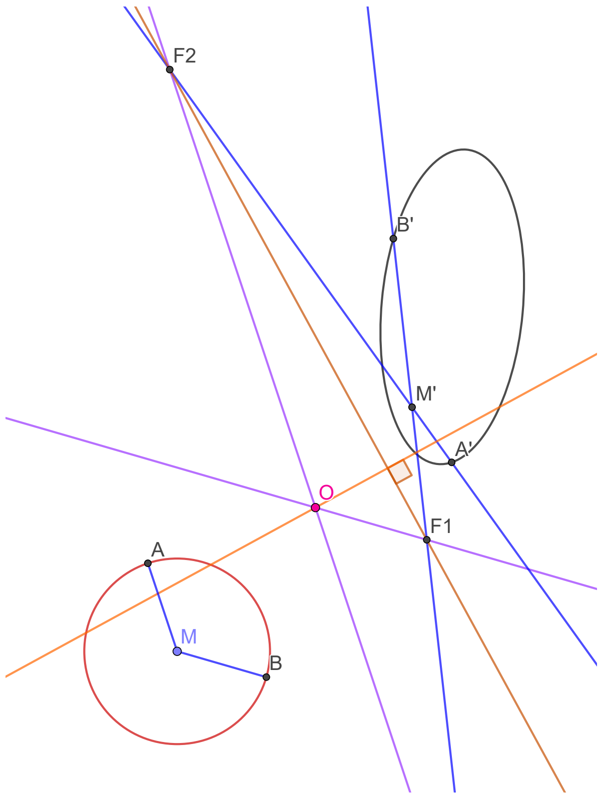

Figure 3: Steps 7-8 (previous construction lines removed for clarity)

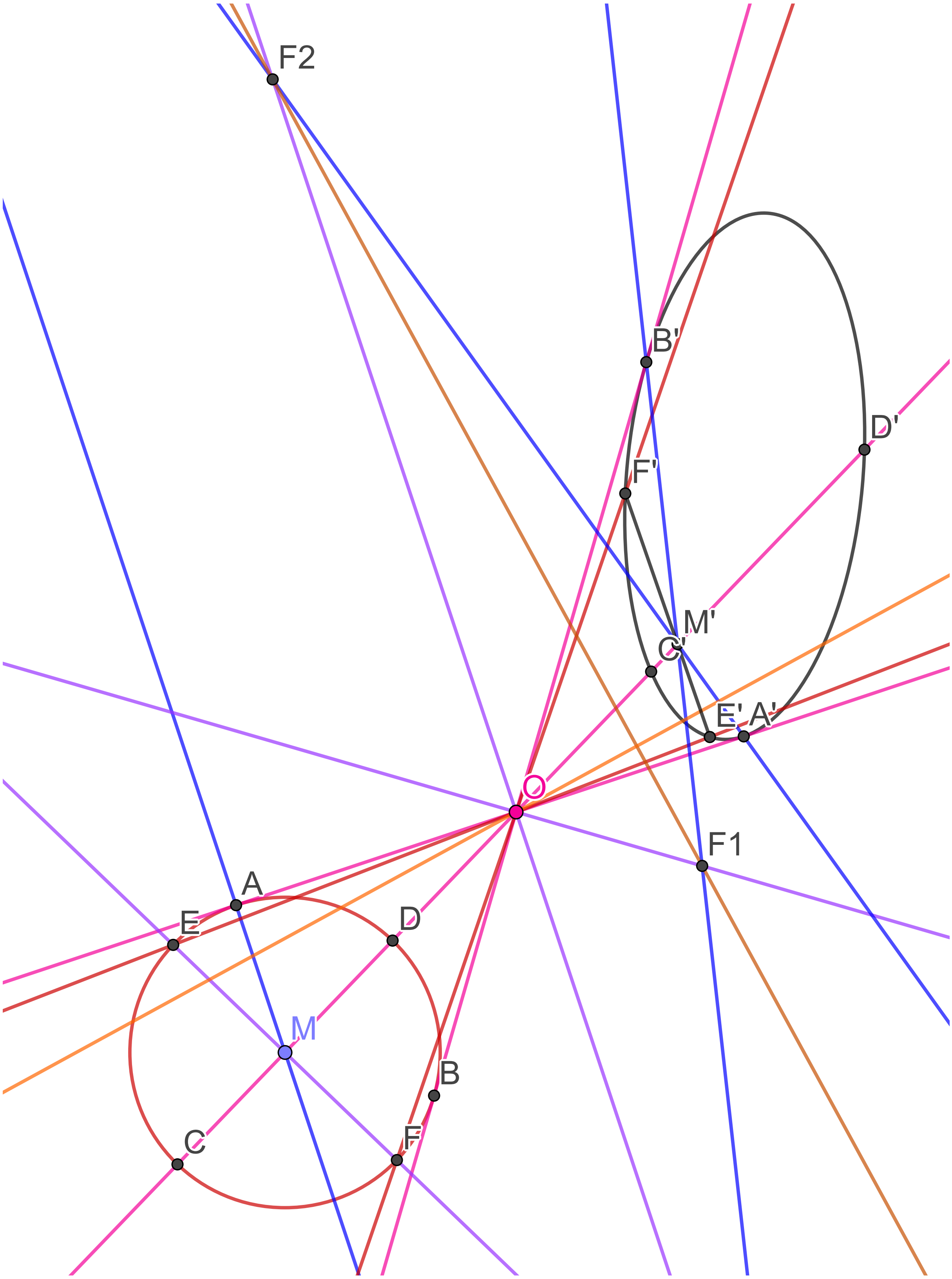

Figure 4: Final graph with all construction lines. Optical axis shown in light orange.