Physics Cup 2025 Problem No.4 Solution
Axis of a lens

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Problem Recap

Geometrically construct (use a ruler and a compass) the main optical axis of the ideal lens from its center and the ellipse image of a circle on the figure shown below.

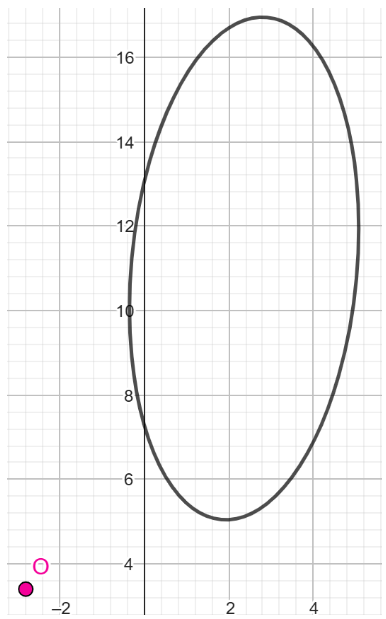

Figure 1: Caption for the image goes here.

Sketch

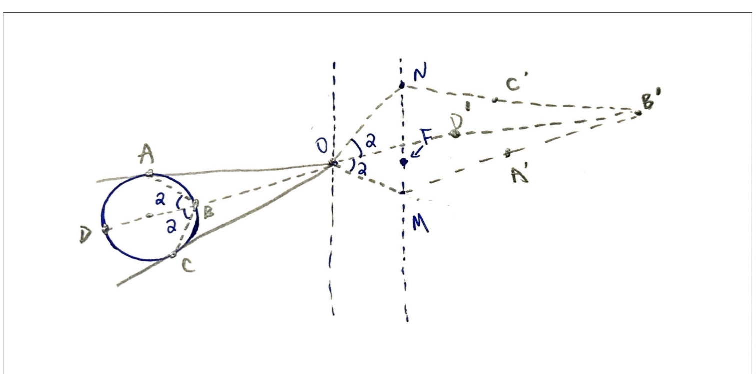

Figure 2: Sketch of the system.

Analysis

The initial step involves constructing the tangent lines to the ellipse from the center O, designated at their intersections as A' and C'. These image points correspond to the real points A and C on the circle. Since optical rays passing through the lens center will maintain their direction, the angle bisectors of $\angle A'OC$ intersect the circle and the ellipse at points B, D and B', D', respectively.

As depicted in Figure 2, there is a one-to-one correspondence between these points:

$$\begin{aligned} A &\to A', \\ B &\to B', \\ C &\to C', \\ D &\to D'. \end{aligned}$$

Denote the angles $\angle ABD$ and $\angle CBD$ by α. The point extending infinitely in the direction of BA, beyond the optical center O, has its image on the focal plane. This implies that lines $B'A'$ and $B'C'$ intersect the focal plane at points M and N, where $\angle NOF = \angle MOF = \alpha$.

The remaining task involves constructing the angle α, which can be approached using various methods. In this solution, I employ the "bisector method," where three distinct bisectors are constructed.

Solution

Steps for Constructing the Main Optical Axis:

1. Begin by constructing tangent lines to the ellipse from the center point O, denoted as lines OF and OH, intersection with the ellipse are F and H.
2. Next, bisect the angle formed between $\angle FOH$. This angle bisector will intersect the ellipse at points G and I.
3. Find the angle bisector of $\angle IOH$, which will be referred to as line l.
4. Additionally, draw line m perpendicular to this bisector.
5. Construct the bisector of the angle formed between lines l and m, known as line p.
6. Determine where line p intersects with line IH; label this intersection point as J.
7. Calculate the symmetrical counterpart of point J about line l, naming it J'.
8. Draw lines OJ' and IF; their intersection generates point K.
9. Connect points K and J; this connection represents the focal plane near the image.
10. Finally, construct a line perpendicular to the focal plane that passes through lens center O. This line is depicted as line s in the figure and serves as the principal axis.

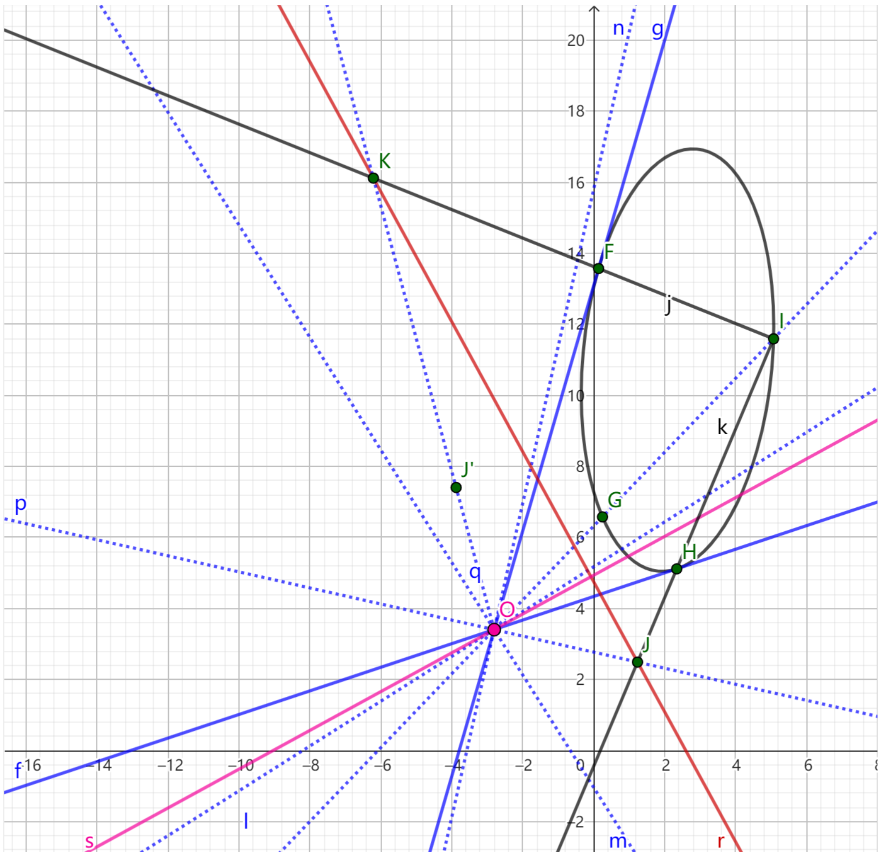

Figure 3: Solution, where the pink line s is the axis of the lens.

Answer

The principle axis of the lens is:

$$y = 0.54627393x + 4.93702565$$

Appendix

I did attempt to construct the lines OM and ON by drawing parallels to lines AB and BC, as illustrated in Figure 4. However, the simplest method to obtain AB and BC involves constructing an inscribed circle for f and g. This approach introduces some inaccuracies when using GeoGebra. Specifically, there will be two intersections between the inscribed circle and g, labeled N and Q. The final result may differ at the eighth decimal place depending on the choice of K. If the second intersection Q is eliminated with a particular selection of parameters, the result aligns with the one previously discussed.

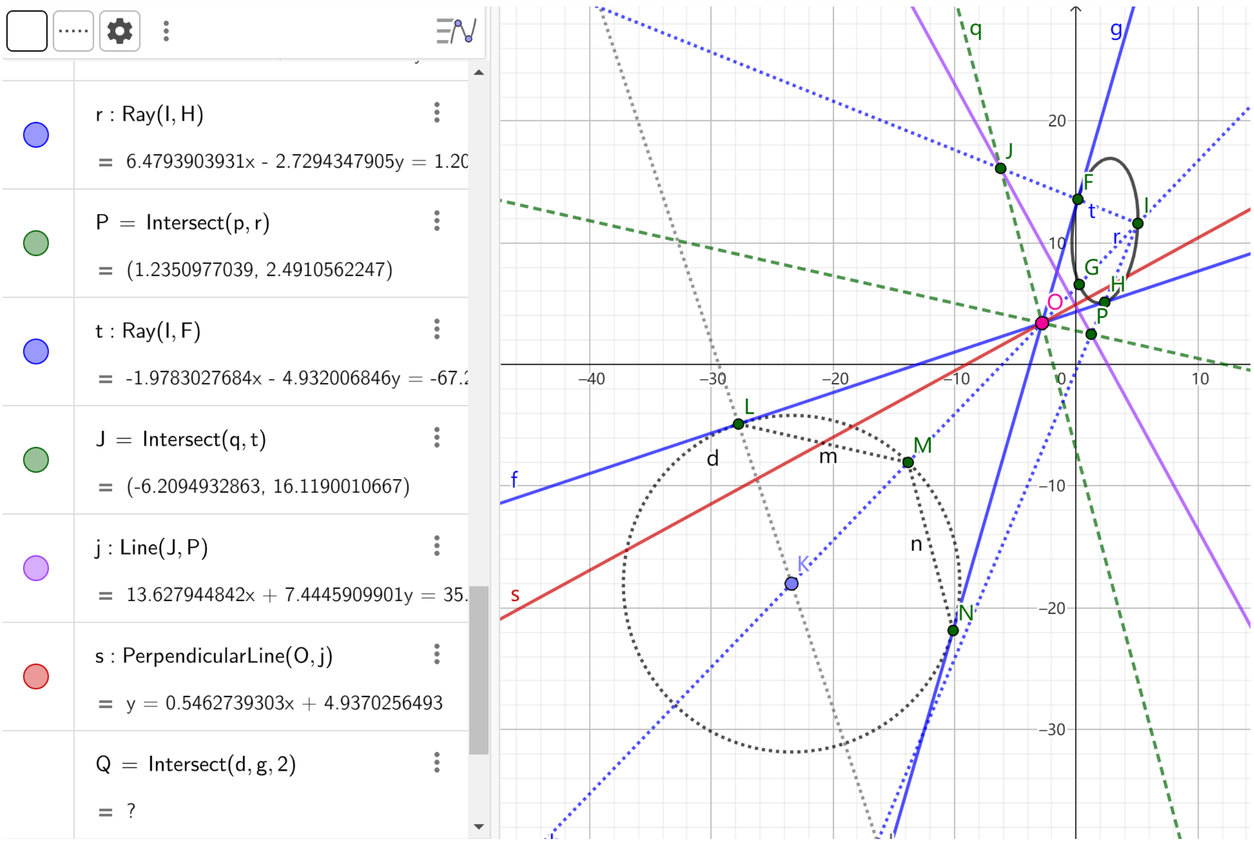

Figure 4: Another method emplying the construction of a inscribed circle.

A better approach involves constructing a perpendicular from L to line OK, and finding its intersection with g, rather than simply determining the intersection points between the circle and g. This method leads to a more accurate result. However, it sacrifices the convenience and simplicity that comes with constructing the inscribed circle.