Physics Cup 2025 - Problem 4

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1 Description of Geometrical Construction

1.1 Tangents to the Ellipse through O

First of all, notice that light rays passing through the centre O of the lens are not deflected. Furthermore, rays tangent to the image are also tangent to the object. Combining these two facts, a tangent through O to the ellipse is a tangent through O to the initial circle. These two tangents are denoted by t_1 and t_2 in the figure. They intersect with the ellipse in A and B respectively.

1.2 Angle Bisector of t_1 and t_2 and Position of the Centre of the Circle

The circle is tangent to t_1 and t_2 as explained above which means that the lines perpendicular to t_1 and t_2 and going through the tangency points (A' and B' respectively) are diameters of the circle. They intersect in the centre C' of the circle. However, since $A'C' = B'C'$, C' is at equal distances from t_1 and t_2 and thus belongs to the angle bisector b of the tangents by the definition of angle bisectors.

As is shown below, the exact position of C' on b isn't important for the problem of finding the optical axis. Thus, C' is placed anywhere on b to the left of O and the circle is constructed by drawing the perpendicular lines (p_1 and p_2) to the tangents through C' to find A' and B'. This circle is not the real circle which gives the ellipse as image, but some circle for which the proportions are the same which is enough to find the optical axis. In the following, it is assumed that this circle is the real one, because as is shown below, displacing C' doesn't change the answer.

The angle bisector b intersects with the circle in M' and N'. Since a ray on b passes through O, it is not deflected by the lens and thus, M and N, which are the images of M' and N' respectively, are the intersection points of b with the ellipse, as is shown in the figure.

1.3 Construction of the Optical Axis

Since lines stay lines after deflection by the lens, a ray passing through two points also passes through their images. Since the ray is deflected in a point situated on the ideal thin lens, the intersection point of the two lines passing through two points and through their images respectively lies on the lens. Thus, two points of the lens can be found by intersecting the lines AB and $A'B'$ in I_1 and the lines BN and $B'N'$ in I_2. The line l representing the ideal thin lens passes through I_1 and I_2. The optical axis, represented by line a, is the perpendicular to l passing through the centre O of the lens.

1.4 Reason for Independence of Result on Exact Position of C'

Since $B'C' = C'N'$ (they are radii) and since the directions of $B'C'$ and $C'N'$ are fixed, $B'N'$ will have the same direction no matter where C' is placed. For analog reasons, $A'B'$ is fixed in direction. t_2, AB and BN are fixed anyway.

This allows to conclude that the triangles $BB'I_1$ for different positions of C' are similar. Thus, the distance $B'I_1$ is proportional to the distance BB'. In the same way, the triangles $BB'I_2$ for different positions of C' are similar. Thus, the distance $B'I_2$ is proportional to the distance BB'. Since $B'I_1 \propto BB'$ and $B'I_2 \propto BB'$, $B'I_1 \propto B'I_2$. Together with the fact that the directions of $B'N'$ and $A'B'$ are fixed, this allows to conclude that the triangles $B'I_1I_2$ for different positions of C' are similar and that I_1I_2 is constant in direction. Taking the perpendicular to I_1I_2 through O gives the optical axis.

This shows that the arguments above can be made assuming that C' is the real centre of the circle without changing the final result. One can imagine that the construction was done with the real but unknown C' and shifted afterwards.

Note that this is **no trial and error** approach since the constructed optical axis is identical for any such C'. The construction is exact and one does **not** have to look for some special position of C' on b. So trying different positions of C' is **not** necessary.

2 Procedure for Constructing the Optical Axis

- Draw the tangents t_1 and t_2 to the ellipse through the centre O of the lens. Note A and B the points of tangency.
- Draw the angle bisector b of the acute angle formed by the tangents and note the intersections with the ellipse M and N.
- Place a point C' anywhere on b to the left of O. Draw the perpendicular lines p_1 and p_2 to the tangents passing through C' and note the intersections with the tangents A' and B' respectively. Add a circle of centre C' including A' and B' and note the intersections with b as M' and N'.
- Intersect the lines AB and $A'B'$ in I_1 and intersect the lines BN and $B'N'$ in I_2.
- The optical axis a is the perpendicular to I_1I_2 passing through O.

3 Result

The equation for the optical axis is (rounded to 8 decimal digits):

$$\boxed{a : y = 0.54627393x + 4.93702565}$$

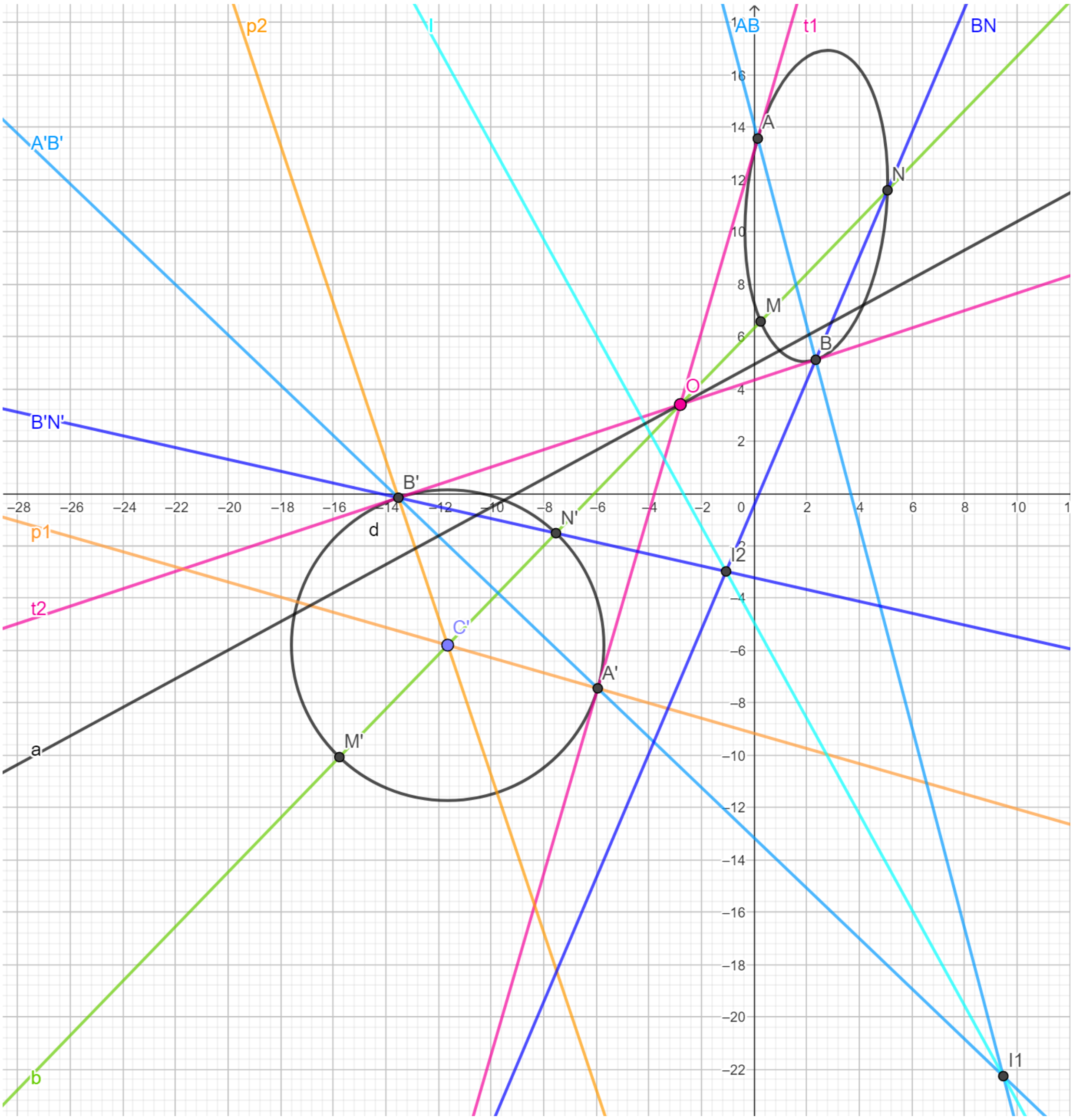

Figure 1: *Construction in GeoGebra*