Physics Cup Problem 4
Dhruv Shah

1. We first find two points on the focal plane.

Step 1 Draw 2 tangents t_1, t_2 from the center of the lens, O, to the ellipse Ω. These are tangent at points P_1 and P_2 resp.

Step 2 Draw the angle bisector u, of t_1 and t_2. Label the points Q_1, Q_2 where $u \cap \Omega = \{Q_1, Q_2\}$.

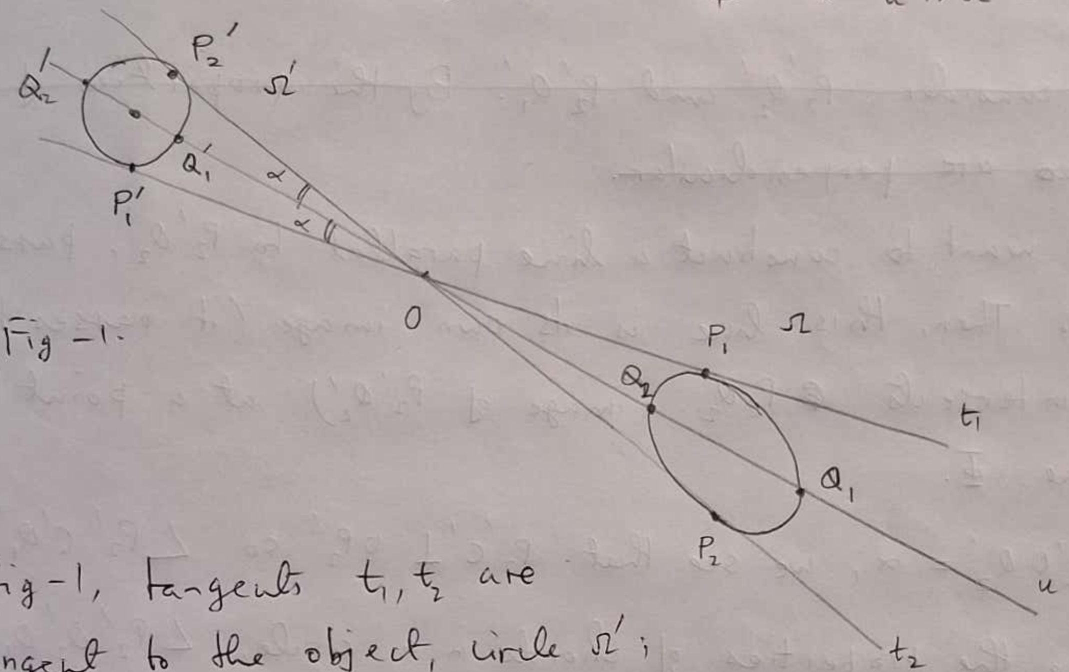

Fig -1.

From fig-1, tangents t_1, t_2 are also tangent to the object, circle Ω'; at points P_1', P_2' resp. Clearly, the center of circle Ω' must lie on u. Let $\angle P_1 O Q_1 = \alpha$.

2. Now, we have a set of parallel lines in the object space, i.e.
(1) tangent to Ω' at Q_1': v_1' (2) tangent to Ω' at Q_2' : v_2'
(3) line $P_1'P_2'$.
(We can show that these lines are parallel by simple facts about tangents, and some angle chasing). Further, all the 3 lines are perpendicular to u.

- We use the fact that the images of parallel lines intersect in the focal plane Φ.

Step 3 Draw tangents to ellipse Ω at points Q_1, Q_2; label these

as v_1, v_2. Label the point $v_1 \cap v_2 = \{J\}$. Note that $J \in \Phi$.
One can also check that P_1, P_2 and J are collinear (as it should be, given $P_1'P_2' \parallel v_1' \parallel v_2'$).

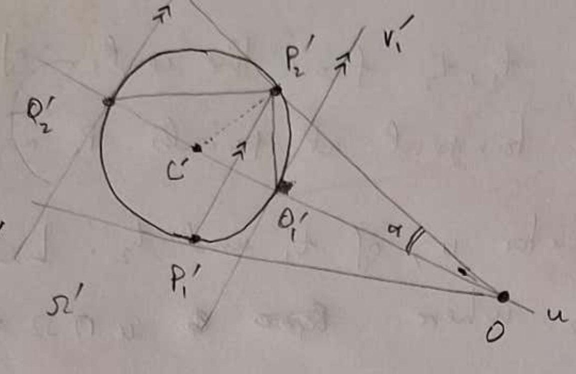

Fig-2.

3. ~~Sto~~ ~~Now consider $P_2'Q_2'$ and $P_2'Q_1'$. By the properties of a circle, these lines are perpendicular.~~

Now, we want to construct a line parallel to $P_2'Q_2'$, passing through O. Then, this line is its own image (it passes through O), and it intersects P_2Q_2 (image of $P_2'Q_2'$) at a point on the focal plane Φ.

With $\angle P_2'OQ_2' = \alpha$, we see that $P_2'C' \perp OP_2'$ so $\angle P_2'C'Q_1' = 90° - \alpha$.

So, using the properties of chords in a circle, $\angle P_2'Q_2'Q_1' = \frac{90° - \alpha}{2}$.

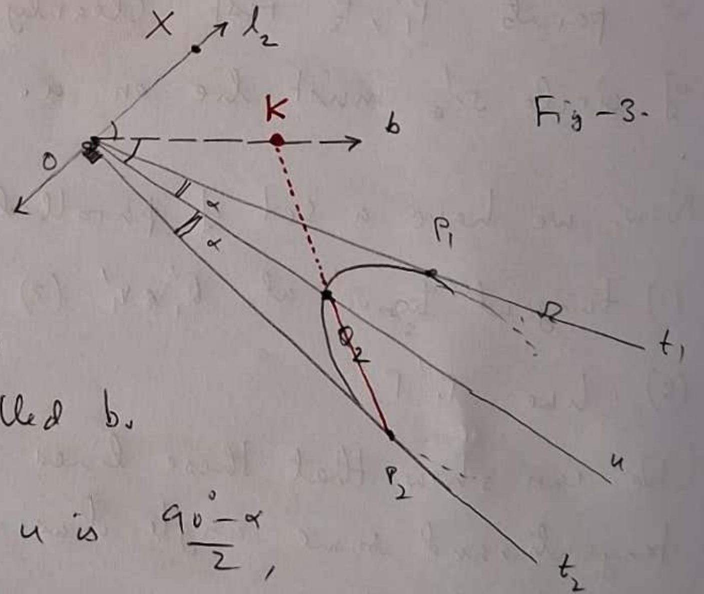

Fig-3.

$\boxed{\text{Step 4}}$ Construct line $l_2 \perp t_2$ through O. For some $X \in l_2$, $\angle XOQ_1 = 90° - \alpha$.

$\boxed{\text{Step 5}}$ Bisect angle $\angle XOQ_1$. The angle bisector line is called b.

- Since the angle between b and u is $\frac{90° - \alpha}{2}$, $b \parallel P_2'Q_2'$.

$\boxed{\text{Step 6}}$ Extend P_2Q_2 to line b. The point of intersection is called K.

From previous reasoning, $K \in \Phi$.

4\. $\boxed{\text{Step 7}}$ Construct line JK. This is a 2D projection of the focal plane. Finally, construct a line $\perp$ to JK, passing thr' O.
This is the axis of the lens.

Equation: $y = 0.5462739303x + 4.9370256493$.