Physics Cup 2025 Problem 4

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Motivation

To motivate our Geogebra construction we will use the following important properties of thin lenses:

1. Rays passing through the center O will not be refracted.
2. Rays tangent to the object will be tangent to the image as well, since they must pass through exactly one point of the object as well as its image.
3. Parallel rays will converge at a point on the focal plane. This is because the parallel rays appear to come from an object at infinity, and by the thin lens equation the image of this object (i.e. where the rays converge) must be in the focal plane.

Property 1 will be used to construct lines that contain important points on the circle. We will combine Properties 2 and 3 and utilize tangents to the circle that are parallel to each other; these will come from diametrically opposite points.

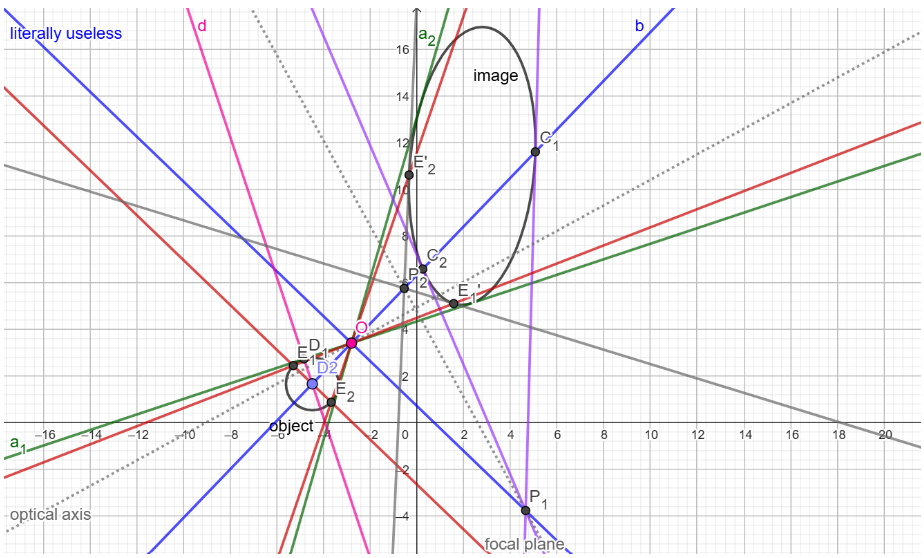
The plan is as follows: We will first find tangents to the circle, and by symmetry we can take their angle bisector to determine the line that the circle's center lies in. We then need to find two pairs of diametrically opposite points on the circle (one pair will come from the angle bisector mentioned earlier). Their tangents will be parallel to each other, and hence, if we can find their image points, we can take the tangents of the image points and mark where they intersect. The intersection point will lie on the focal plane. If we obtain two points of intersection from two pairs of tangents, we can construct the focal plane and subsequently the optical axis.

Construction

The following table shows the steps for construction. The notation used follows the notation in the Geogebra file (also found in the following page).

Step	Construction	Reasoning
1.	Construct two tangents to the image passing through O ($\mathbf{a}_1$ and $\mathbf{a}_2$).	By Properties 1 and 2, these lines are also tangent to the circle.
2.	Construct the angle bisectors of $\mathbf{a}_1$ and $\mathbf{a}_2$. We only care about the line that also passes through the image (**b**).	By symmetry, the center of the circle must lie on **b**.
3.	Mark the intersection points of b with the image as C_1 and C_2. Construct tangents to the image passing through these two points and mark their intersection as P_1.	C_1 and C_2 are the images of the diametrically opposite points of the circle that lie on **b**. As mentioned before, P_1 lies in the focal plane.
4.	Construct a line perpendicular to $\mathbf{a}_1$ (**d**) passing through any point. Mark the intersection of $\mathbf{a}_1$ and **d** as point D_1 and the intersection of **b** and **d** as point D_2. Construct a circle with center D_2 that passes through D_1.	The circle constructed will be tangent to $\mathbf{a}_1$ and $\mathbf{a}_2$. Note that the position of the circle does not matter. Due to symmetry, for a given point X on the circle such that angle D_1D_2X is a constant no matter where the circle is, the angle XOD_1 is also constant,so the line XO always represents a ray coming from point X.
5.	Construct a line perpendicular to **b** and mark its intersection point with the circle as E_1 and E_2. Construct lines E_1O and E_2O and mark their closer-to-O intersection points with the image as $E_1^{'}$ and $E_2^{'}$.	This creates another pair of diametrically opposite points. Note that E_1O and E_2O each pass through two points on the object; from the thin lens equation, since E_1 and E_2 are the further-from O points on the object, $E_1^{'}$ and $E_2^{'}$ must be the closer-to-O points on the image. Note that E_1O and E_2O will always be the same line no matter where the circle was initially constructed due to symmetry.
6.	Construct tangents to the image passing through $E_1^{'}$ and $E_2^{'}$ and record their intersection point as P_2. Construct P_1P_2 (the focal plane), then construct a line perpendicular to P_1P_2 that passes through O (the optical axis).	As mentioned earlier, P_2 lies in the focal plane, as does P_1, hence P_1P_2 is the focal plane. The optical axis is, of course, perpendicular to the focal plane and also passes through O.

A screenshot of the Geogebra construction is attached below. Step 1 is done in green, Step 2 in blue, Step 3 in violet, Step 4 in pink, Step 5 in red and Step 6 in gray. The focal plane and optical axis are presented as dotted lines for ease of recognition. Note that the circle is **not** in the correct place; it was only constructed to help us find the optical axis, and we were not asked to determine the location of the circle. Unnecessary labels are removed and irrelevant lines are labelled for clarity.



At the end, we obtain an equation for the optical axis in the form of $y = ax + b$, where $a = 0.54627393$ and $b = 4.93702565$ (both to 8 decimal places).