Physics Cup 2025 – Axis of a lens

Francesco Gariglio

Indications about notation and labels

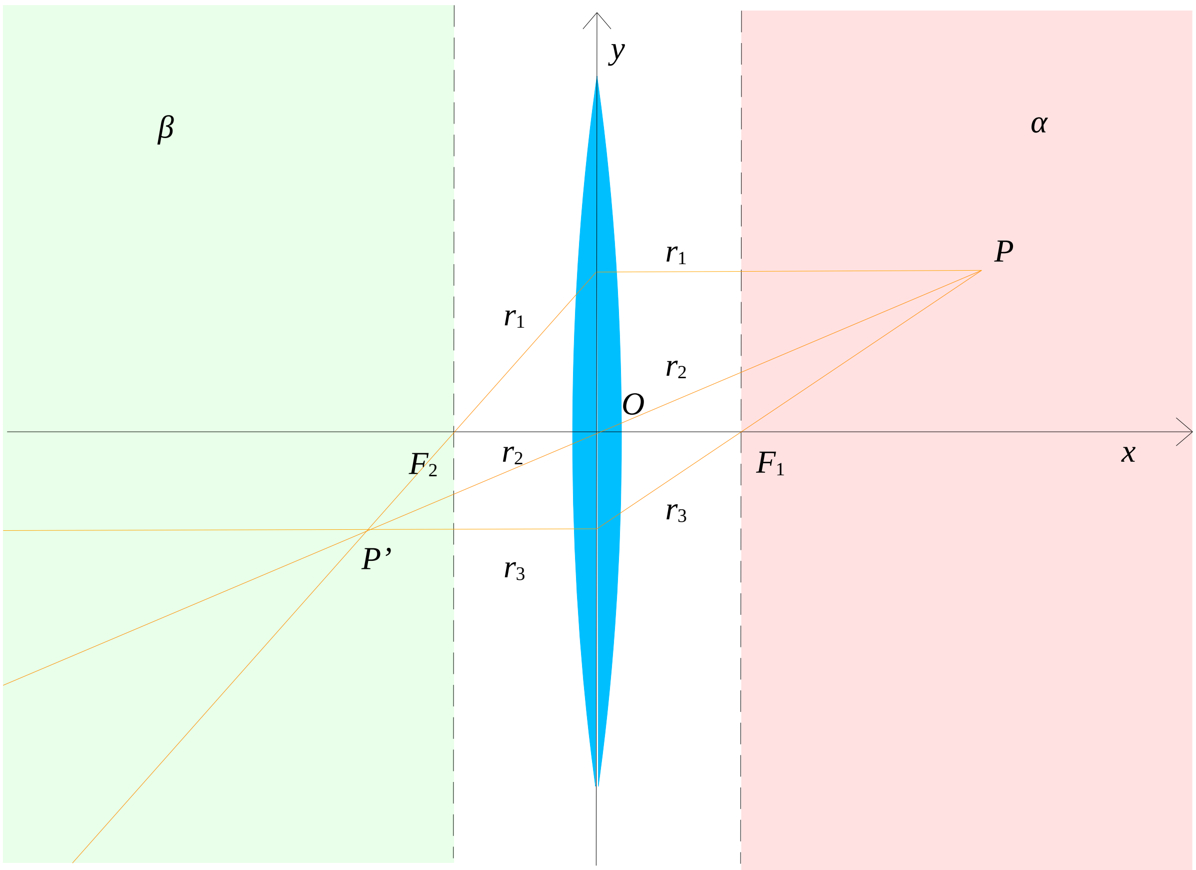
Throughout the text (but not in the GeoGebra file), every letter with an accent indicates the image of the object labeled with the same letter; e.g., P' is the image of point P, while curve γ' is the image of curve γ. This notation will be particularly used in the enunciation of theorems and corollaries.
A "horizontal" straight line is always to be intended as a line parallel to the optical axis of the lens, while a "vertical" straight line is always to be intended as a line orthogonal to the optical axis.
"Ray" means a "light ray", and not a half-line, which will be named "half-line".

Preliminary theorems

Theorem I: *there is a one-to-one correspondence between points and their own image points.*

Let O be the center of an ideal thin lens. Let us take a Cartesian coordinate system in which O is the origin and the x-axis coincides with the optical axis of the lens, as shown in *figure 1*:

figure 1



We will study only the two semi-planes highlighted, as we do not have to consider virtual images.
Let x_P, y_P be the coordinates of point P; it is known that the image point P' will have coordinates

$$\left(\frac{x_P f}{f-x_P}, \frac{y_P f}{f-x_P}\right),$$

being f the length of segment OF_1.
Since the two functions which relate x and y coordinates of P and P' are bijective, it is evident that there must be a one-to-one correspondence between points belonging to semi-plane α and points belonging to semi-plane β.

Corollary I: *if two curves* γ_1 *and* γ_2 *intersect in and only in* n *points* $A_1, A_2, \dots, A_n$, *then* γ_1' *and* γ_2' *will intersect in and only in the* n *points* $A_1', A_2', \dots$ *and* A_n'.
Corollary I_{bis}*:* *if two curves are mutually tangent in and only in* n *points* $A_1, A_2, \dots, A_n$, *then* γ_1' *and* γ_2' *will be tangent in and only in* n *points* $A_1', A_2', \dots$ *and* A_n'.
Corollary II: *if a curve* γ *passes through* n *points* $A_1, A_2, \dots, A_n$, *then* γ' *will pass through* $A_1', A_2', \dots$ *and* A_n.

Theorem II: *the image of a segment or of a half-line lying in* α *(see* figure 1*) is a segment or a half-line lying in* β.

In a subset of a straight line, for three general point $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ belonging to it the relation

$$\frac{y_2-y_1}{x_2-x_1} = \frac{y_3-y_2}{x_3-x_2}$$

holds.
For the corresponding image points, the same relation holds, as we have

$$\frac{\dfrac{y_2 f}{f-x_2}-\dfrac{y_1 f}{f-x_1}}{\dfrac{x_2 f}{f-x_2}-\dfrac{x_1 f}{f-x_1}} = \frac{y_1 x_2-y_2 x_1+f(y_2-y_1)}{f(x_2-x_1)} = \frac{\dfrac{y_3 f}{f-x_3}-\dfrac{y_2 f}{f-x_2}}{\dfrac{x_3 f}{f-x_3}-\dfrac{x_2 f}{f-x_2}} = \frac{y_2 x_3-y_3 x_2+f(y_3-y_2)}{f(x_3-x_2)}.$$

In fact, the substitution of y_i with $mx_i + q$, which implies $y_1x_2 - y_2x_1 = q(x_2 - x_1)$, leads to

$$\frac{q}{f}+\frac{y_2-y_1}{x_2-x_1} = \frac{q}{f}+\frac{y_3-y_2}{x_3-x_2}.$$

This is the necessary and sufficient condition to have three points collinear; the theorem is proved. Moreover, for *corollary II*, the line on which lies the image of the subset of the straight line is determined if we know at least two points belonging to it.

Theorem III: *the image of a vertical straight line is a vertical straight line.*

A vertical straight line is characterized by an equation of the form $x = k$. This means that the image of each arbitrary point belonging to this line will have an abscissa equal to $\frac{kf}{f-k}$, and, since there is a one-to-one correspondence, the image of the vertical line must be the vertical line of equation

$$x = \frac{kf}{f-k}.$$

Theorem IV: *the image of a horizontal straight half-line* s *in semi-plane* α *(see* figure 1*) lies on a straight line passing through* F_2.

Let *P* be a point belonging to *s*. *P'* can be geometrically found as the intersection between the rays labeled r_1 and r_2 (*figure 1*). By construction, r_1 is initially parallel to the optical axis (viz. it is horizontal, thus it coincides with *s*), and, after being refracted by the lens, it passes through F_2. However, since, after the refraction, r_1 covers the same path for each point belonging to *s*, it implies that *P'*, which lies at the intersection of r_1 and r_2, must lie on a line passing through F_2.

Theorem V: *the image of a subset of a straight line* s *passing through* O, *this subset being entirely in semi-plane* α *(see* figure 1*), lies on the same straight line* s.

Let *P* be a point belonging to *s*. *P'* can be geometrically found as the intersection between the rays labeled r_1 and r_2 (*figure 1*). By construction, r_2 coincides with *s*, for each point of *s*, as they both pass through *P* and *O*. Since *P'* lies at the intersection of r_1 and r_2, it must lie on the same straight line *s*.

Theorem VI: *the images of two parallel straight half-lines lying in α (see* figure 1*) lie on two lines which intersection lies in the vertical line passing through* F_2.

It has been proved in *theorem II* that the image of a subset, lying in α, of line s: $y = mx + q$ lies in β, on a line of slope $\frac{q}{f}+m$; it can be easily proved that the y-intercept remain q also for the image line. Thus, the lens transform a half-line lying on line $y = mx + q$ in a segment or a half-line lying on line $y = \left(\frac{q}{f}+m\right)x + q$.

Two parallel half-lines lie on lines which can be expressed by equations

$$y = mx + q_1$$
$$y = mx + q_2;$$

their images lies on

$$y = \left(\frac{q_1}{f}+m\right)x + q_1$$
$$y = \left(\frac{q_2}{f}+m\right)x + q_2$$

It can be easily proved that the intersection of these two lines has got an abscissa equal to $-f$.
A less rigorous proof of the theorem is the following: two parallel half-lines can be regarded as intersecting for $x \to \infty$; thus, for *corollary I*, their images will meet in a point lying on the vertical line passing through F_2.

Geometrical construction of the optical axis

We will find the focus of the lens as the intersection of two conics.
Two properties of the circle will be analyzed, shown in *figure 2* and *figure 3*:

figure 2

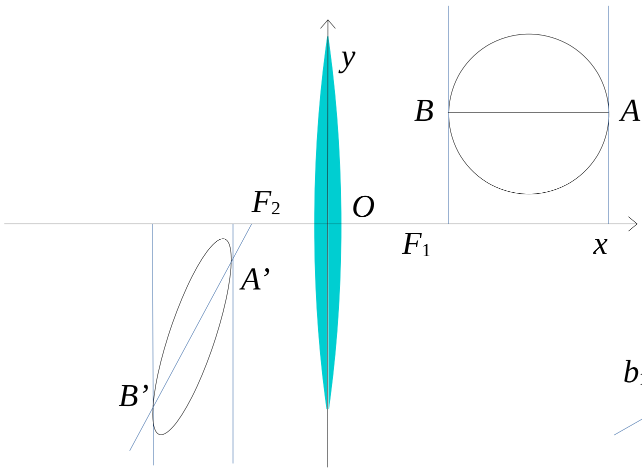
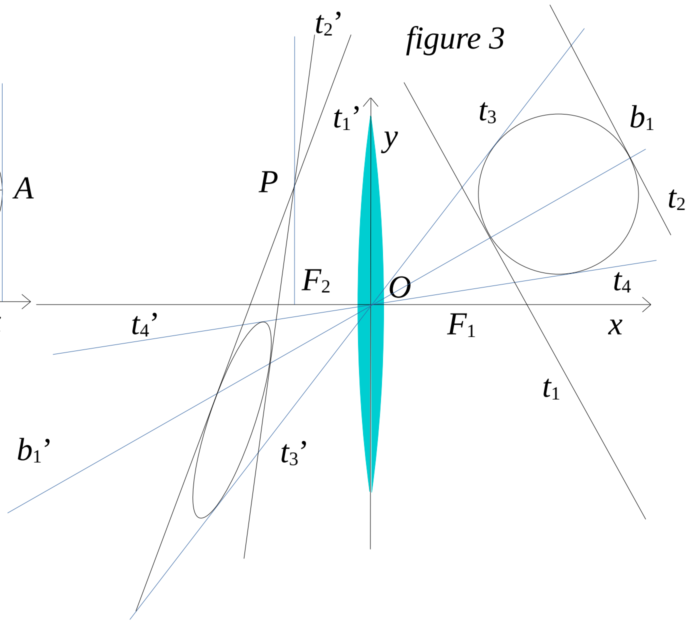


figure 3



In *figure 2*, it can be observed that the segment AB is parallel to the optical axis. A and B are the points of tangency of the circle with the lines of the improper beam perpendicular to the optical axis. For *corollary* I_{bis} and *theorems III* and *IV*, A', B' and F_2 must be collinear. Since we can find A' and B' as the points of tangency of the given image ellipse with two lines perpendicular to the optical axis, the position of F_2 is determined after that an optical axis is chosen. In the appendix it is proved that, by this condition, we find that **F_2 must lie on an equilateral hyperbola, labeled γ_1**.
In *figure 3*, it can be observed that $t_1 \parallel t_2$. In fact, since b_1 is, by construction, the line bisecting tangents t_3 and t_4, it passes through the center of the circle; therefore, tangents t_1 and t_2 are parallel. For *corollaries I* and I_{bis} and *theorems V* and *VI*, the intersection of the line orthogonal to the optical axis which passes through P and the optical axis itself is the focus F_2. Since segments PF_2 and OF_2 are always orthogonal by construction, **F_2 must lie on a circumference of diameter OP, labeled γ_2**.
Point P can be easily found following these steps: trace the two tangents t_3' and t_4' to the image ellipse passing through O, and then trace the line b_1' bisecting them. Find the two intersections between this bisector and the ellipse, then, for each intersection, trace the tangents to the ellipse which passes through the intersection. P is the intersection of the last two tangents, labeled in *figure 3* t_1' and t_2'.
If we want the ellipse to be the image of a circle, these two conditions are necessary, and they both must be fulfilled. It follows that F_2 must lie in one of the intersections γ_1 and γ_2.
We get four intersections; however, as we know that the image of the circle is real, only one is acceptable, the one named F_2 in the GeoGebra file. The others, since the vertical line (viz. the line orthogonal to the optical axis) passing through O intersect the image ellipse, are not acceptable, as the image would be divided in two parts by the lens. This is not compatible with the condition that the ellipse is the image of a single circle.
It is now possible to draw the line passing through O and F_2, and so we obtain the optical axis and its equation:

$$y = 0{,}54627393x + 4{,}93702565.$$

The coefficient requested a is therefore equal to 0,54627393, rounded down at the eighth decimal digit.

Appendix

Let us consider an ellipse of semi-axis a and b ($a > b$) and a point $O(x_O, y_O)$. One can take a Cartesian system so that the center of the ellipse is the origin and the x-axis coincides with the major semi-axis of the conic. It is now possible to dilate the x-component of the position vectors by a factor b/a; after the dilatation, we get a circle of radius b and a point $O_1(x_{O1}, y_O)$. Let us define line s_1:

$$s_1: y - y_O = \frac{a}{b} m(x - x_{O1});$$

it represents the optical axis after the dilatation.
We now want to determine the slope of line t_1 so that, after a new dilatation along x-axis of factor a/b, t and s are perpendicular, being t and s the lines t_1 and s_1 after the second dilatation. Two lines are perpendicular if and only if the product of their slopes gives – 1; if we dilate the same lines of a factor b/a, the product of the new slopes gives $- a^2/b^2$. Hence, the slope of t_1 must be $-\frac{a}{bm}$. We have to take the two points of tangency of the dilated ellipse (viz. the circle) with the improper beam of straight lines of slope $-\frac{a}{bm}$. Line u_1, passing through this two points, passes through the origin of the coordinate system, and it is orthogonal to the two tangents (for the properties of the circumference); it must have equation $y = \frac{bm}{a}x$. Finally, we can dilate all of a factor a/b, to return in the initial situation; focus F_2 coincide with the intersection of s and u, of equations

$$s: y = m(x - x_O) + y_O$$
$$u: y = \frac{b^2 m}{a^2} x.$$

For the ordinate and the abscissa of their intersection (which is F_2) the relation

$$y = \frac{-y_O x}{\left(\frac{a^2}{b^2} - 1\right) x - \frac{a^2}{b^2} x_O}$$

holds.
This means that F_2 must lie on an equilateral hyperbola which passes through the center of the given ellipse and through O. We can determine three other points belonging to the hyperbola by tracing three random optical axis and by geometrically determine the intersection between the optical axis and the line passing through the two points of tangency, as shown in *figure 2*. When at least five points belonging to the hyperbola are known, we can trace the conic.