Physics Cup – TalTech 2025

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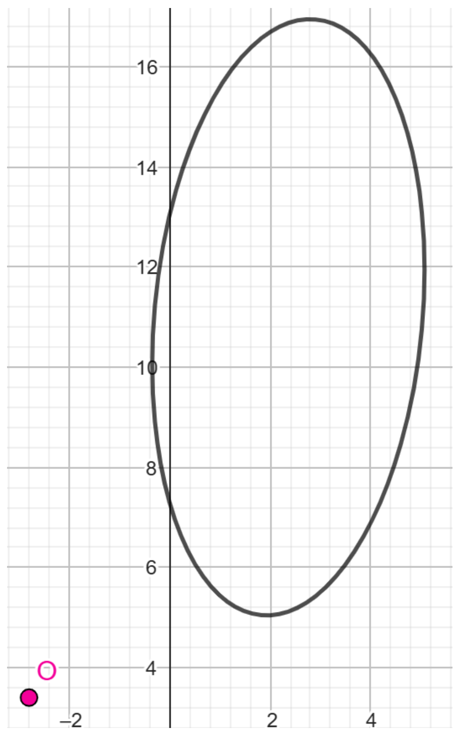
1 Problem 4: Axis of a lens

In the figure below, the ellipse is a real image of a circle created by an ideal thin lens whose center is at point O. Geometrically construct the main optical axis of this thin lens, which is known to lie in the plane of this figure.



2 Facts

Here, for clarity and structure, I will present and prove some facts needed for the construction. Note that in **Fact 1** and **Fact 2**, the coordinate x is along the optical axis, pointing from the object to the image, with $x = 0$ at the lens center. **Remark:** For objects $x < 0$, and for images $x > 0$.

Fact 1. *If two points have x-coordinates x_1 and x_2, and their image x-coordinates are x'_1 and x'_2, respectively; then*

$$x_1 < x_2 \Leftrightarrow x'_1 < x'_2$$

.

Proof. According to the lens equation, $\frac{1}{f} = \frac{1}{x'} - \frac{1}{x}$, which implies that $x = \frac{x' f}{f - x'}$ and $x' = \frac{x(-f)}{(-f) - x}$. In the forward direction, $x_1 < x_2$ and the lens equation implies that $\frac{x'_1 f}{f - x'_1} < \frac{x'_2 f}{f - x'_2}$, from which $x'_1 f - x'_1 x'_2 < x'_2 f - x'_1 x'_2$ follows. Finally, we arrive at $x'_1 < x'_2$. In the reverse direction, the proof is trivial by interchanging $x \leftrightarrow x'$ and $f \leftrightarrow -f$ and repeating the forward direction steps. This completes the proof. □

Fact 2. *Let l_1 and l_2 be two non-parallel lines intersecting at point O. Let points P and Q lie on l_1 such that $PO < PQ$, and let $H \in l_2$ be such that $\overleftrightarrow{PH} \perp l_1$. Define lines f_1, f_2, g_1, and g_2 to be parallel to some fixed lines f_1^*, f_2^*, g_1^*, and g_2^* respectively, such that $f_1 \cap f_2 = P$ and $g_1 \cap g_2 = Q$. Let $A = f_1 \cap g_1$ and $B = f_2 \cap g_2$. Then, if $OH = s$ and $\angle HOP = \alpha$, we have $\vec{AB} = h(s)\vec{v}$ for some vector $\vec{v}$ independent of s and a function h of s.*[1]

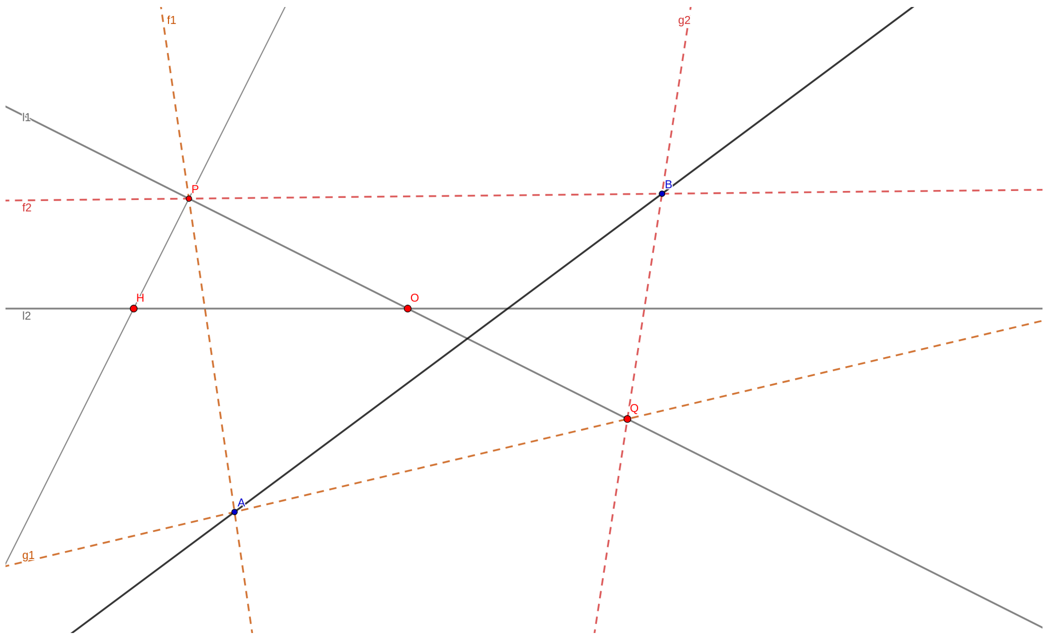


Figure 1: A particular case.

Proof. Setting our coordinate system such that $Q = (0,0)$ and the x-axis along l_1, it is straightforward to see that if $QO = d$, then the length $l = QP = d + s\cos\alpha = h(s)$. Let k_1, k_2, m_1, and m_2 be the slopes of lines f_1, f_2, g_1, and g_2 respectively, which are independent of s because they remain parallel to fixed lines. Hence, the equations of the lines are given by $x_f = ky_f - l$ and $x_g = my_g$. By equating, we obtain the intersection point at $ky_0 - l = my_0$, which gives us the intersection point $I = \left(\frac{ml}{k-m}, \frac{l}{k-m}\right)$. From here, we can construct the vector $\vec{AB}$ which is given by: $\vec{AB} = l\left(\frac{m_1}{k_2-m_2} - \frac{m_2}{k_1-m_1}, \frac{1}{k_2-m_2} - \frac{1}{k_1-m_1}\right)$. Therefore, $\vec{AB} = h(s)\ \vec{v}(k_1, k_2, m_1, m_2)$, where $\vec{v}$ is independent of s as desired. □

Fact 3. *A segment object generates a segment image in such a way that their projections intersect at the lens axis (vertical axis).*

Proof. It is sufficient to prove that the point of intersection is located at $x = 0$. The projection line of the object can be written as $y = ax + b$. Picking an object point, by the lens equation, $x = \frac{x'f}{f-x'}$ and $y = \frac{y'f}{f-x'}$[2]. Replacing these into the line equation, we get:

$$\frac{y'f}{f-x'} = k\frac{x'f}{f-x'} + b$$

$$y' = \left(k - \frac{b}{f}\right)x' + b$$

Therefore, the image is also a line segment, and notice the intercept of both lines' projections with the y-axis is at $y = b$, from which they intersect at $(0, b)$ as desired. □

[1] This means that the line $\overleftrightarrow{AB}$ remains parallel as the distance s changes.
[2] This is just applying $y/x = y'/x'$ because of the ray that goes through the center of the lens.

3 Optical Axis Construction

The goal is to construct a line formed by intersection points that is perpendicular to the true lens axis. First, I will describe the construction, and then I will prove that it defines the optical axis. Here is a link to the GeoGebra File.

> Consider the ellipse Γ. Let T and T' be points on Γ such that the lines $\overleftrightarrow{OT}$ and $\overleftrightarrow{OT'}$ are tangent to Γ. Define m as the bisector of the acute angle formed by $\overleftrightarrow{OT}$ and $\overleftrightarrow{OT'}$. Let C be an arbitrary point on m such that $\overrightarrow{OC} \cap \Gamma = \emptyset$. Define Σ as a circle centered at C and tangent to $\overleftrightarrow{OT}$ and $\overleftrightarrow{OT'}$ at H, and H', respectively. Let D be the closest intersection point of $\overleftrightarrow{CO}$ with Σ, relative to O. Let E be the intersection point of $\overleftrightarrow{CO}$ and Γ that is furthest from O. Define A as the intersection point of $\overleftrightarrow{TT'}$ and $\overleftrightarrow{HH'}$, and let B be the intersection point of $\overleftrightarrow{TE}$ and $\overleftrightarrow{HD}$. Finally, let P be a point on $\overleftrightarrow{AB}$ such that $\overleftrightarrow{PO} \perp \overleftrightarrow{AB}$. Thus, the line $\overleftrightarrow{PO}$ represents the optical axis.

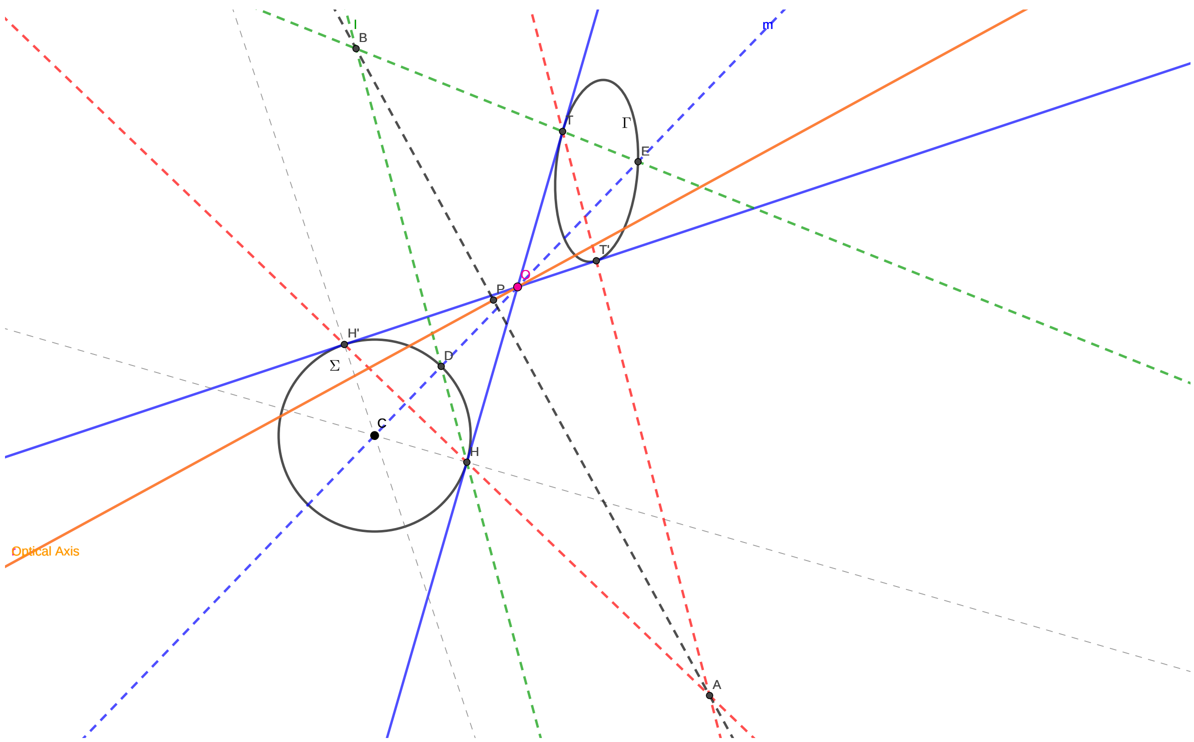


Figure 2: Geometric construction created using GeoGebra.

4 Proof that $\overleftrightarrow{PO}$ is the Optical Axis

From an object point to its image, there exists a ray that passes through O such that it does not deflect. Consider a point outside the angular sector of the acute angle α formed by the lines $\overleftrightarrow{OT}$ and $\overleftrightarrow{OT'}$. This point has no image because no ray passing through O intersects Γ, as all points on Γ are located within the opposite angular sector of α. Hence, the object is contain in the angular sector of α. Now, consider the circular object Ω. Since the lines $\overleftrightarrow{OT}$ and $\overleftrightarrow{OT'}$ intersect Γ at only one point each, the images of T and T', denoted as H_0 and H_0', must lie on the lines $\overleftrightarrow{OT}$ and $\overleftrightarrow{OT'}$, respectively, and are also tangent to Ω. By the Power of a Point Theorem, we have $OH_0 = OH_0'$, which implies that the center C_0 of Ω must lie on the line $\overleftrightarrow{OC}$[3]. Construct D_0, A_0, and B_0 similarly to its non-index points but for Ω. By using the fact that Γ is a real image and **Fact 1**, the image of D_0 is E. Because $\triangle OHH' \sim \triangle OH_0H_0'$ and $\triangle OHD \sim \triangle OH_0D_0$ [4], Thales's theorem implies $\overleftrightarrow{HH'} \parallel \overleftrightarrow{H_0H_0'}$ and $\overleftrightarrow{HD} \parallel \overleftrightarrow{H_0D_0}$. Applying **Fact 2**, we know that $\overleftrightarrow{AB} \parallel \overleftrightarrow{A_0B_0}$. Finally, by **Fact 3**, the optical axis is perpendicular to $\overleftrightarrow{A_0B_0}$, and thus to $\overleftrightarrow{AB}$, which implies that $\overleftrightarrow{PO}$ is the optical axis. Therefore, the result is proven. $\square$

[3]To prove this, note that $\triangle OC_0H_0 \cong \triangle OC_0H_0'$, which implies that $\overleftrightarrow{OC_0}$ is the bisector of α.
[4]Note $\angle HCO = \angle H_0C_0O = 90 - \alpha$ implies $\angle HDO = \angle H_0D_0O = 135 - \alpha/2$.