Solution to Problem 5 - Satellite orbit

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Task

A satellite orbits a planet. At point A, its speed is v_1. At point B, its speed is v_2 and its velocity vector forms a right angle with the velocity vector at point A. At point C, the velocity is exactly opposite to the velocity at point A, with a magnitude of v_3. Find the eccentricity of the orbit. Also determine the exact numerical value of the eccentricity when $v_1 = 1\mathrm{km/s}$, $v_2 = 2\mathrm{km/s}$ and $v_3 = 3\mathrm{km/s}$.

Solution

The main idea for solving this problem is to use the fact that satellite's hodograph is a circle (in other words, path of the velocity vector in $v_x - v_y$ space is circular). Presented below is an almost complete proof of this fact starting from the conserved quantity known as the eccentricity (LRL) vector[1].

Satellite's hodograph is a circle - proof sketch

Eccentricity vector is given by

$$\vec{e} = \frac{1}{GMm}\vec{v}\times\vec{l} - \hat{r} \implies \vec{v}\times\vec{l} = GMm(\vec{e}+\hat{r})$$

where $\vec{l}$ is the angular momentum vector, M and m are the planet and satellite masses respectively and $\hat{r}$ is the unit vector in the radial direction. $\vec{e}$ has magnitude equal to the eccentricity e.

$\vec{e}$ is a constant vector and $\hat{r}$ has constant magnitude so it is clear that vectors $\vec{v}\times\vec{l}$ lie on a circle with radius GMm and a distance $GMme$ from the origin (see Figure 1). $\vec{l}$ is another constant vector (gravitational force is central so angular momentum is conserved) that is perpendicular to the orbital plane.
It can be argued that $\vec{v}$ should be in the same plane as $\vec{e}, \hat{r}$ (velocity is perpendicular to $\vec{v}\times\vec{l}$ and also proportional to its magnitude, $|\vec{v}\times\vec{l}| = vl$) and follow a circular path. Its radius will be $R \equiv GMm/l$ and the circle is a distance $GMme/l = eR$ from the origin O.

[1] This has been a topic of many previous Physics Cup problems, the most recent being P4 in the 2024 edition. The best solutions can be used as reference here, especially in proving that the LRL vector is conserved and it can be related to the eccentricity of the orbit.

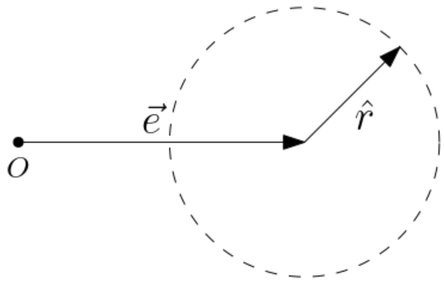

Figure 1: The sum of vectors $\vec{e}$ and $\hat{r}$ lies on a circle

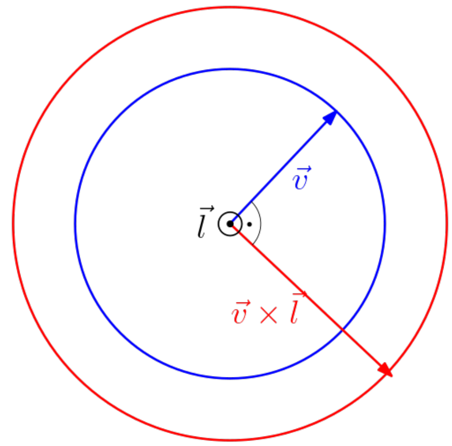

Figure 2: Velocity vector also lies on a circle

Solving for the eccentricity

Choose an arbitrary vector $\vec{v}_1$ in the circle with center S (see Figure 3) on the hodograph. Next, following information in the problem text construct the opposite and perpendicular vectors, $\vec{v}_3$ and $\vec{v}_2$ respectively. Their intersections with the circle are labeled with points K, L and M. Origin is still labeled with O. *Note:* two possible choices for the direction of v_2 both give the same answer for the eccentricity.

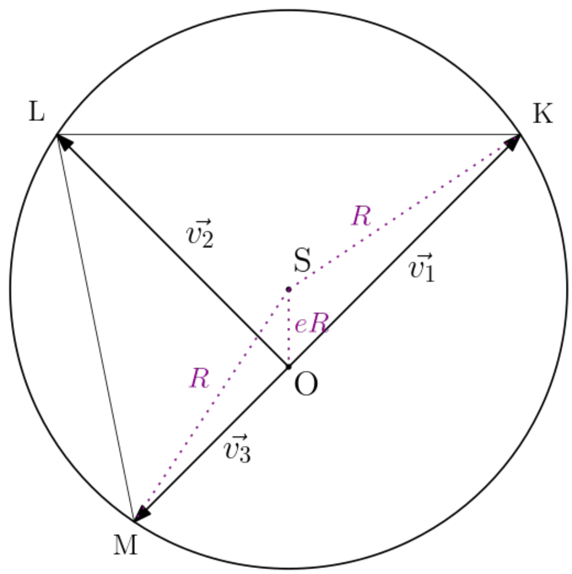

Figure 3: Satellite's hodograph labeled

$\triangle KLM$ is inscribed in a circle of radius R which can be calculated from one of the triangle area formulas:

$$A = \frac{abc}{4R} = \frac{(v_1 + v_3)\sqrt{v_1^2 + v_2^2}\sqrt{v_3^2 + v_2^2}}{4R}$$

On the other hand, area can also be calculated by using the altitude v_2

$$A = \frac{v_2(v_1 + v_3)}{2}$$

from where

$$R = \frac{\sqrt{v_1^2 + v_2^2}\sqrt{v_3^2 + v_2^2}}{2v_2}$$

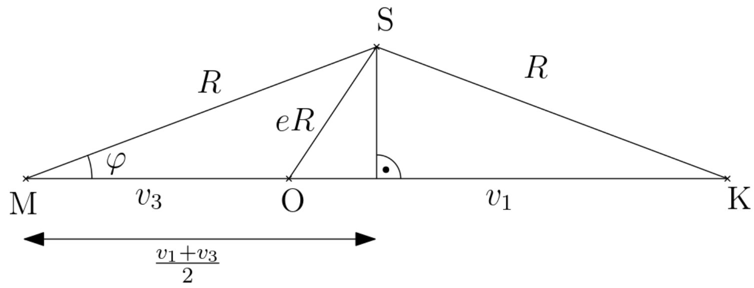

Figure 4: Calculation of eR

eR can be calculated using simple geometry. $\triangle KSM$ is isosceles (two sides of length R and the other v_1+v_3). One of the angles can be determined from a perpendicular triangle (half of $\triangle KSM$, see Figure 4):

$$\cos\varphi = \frac{v_1+v_3}{2R}$$

Using the cosine theorem:

$$eR = \sqrt{R^2 + v_3^2 - 2Rv_3\cos\varphi} = \sqrt{R^2 - v_1 v_3}$$

Substituting R:

$$\boxed{e = \sqrt{1 - \frac{4v_1 v_3 v_2^2}{(v_1^2+v_2^2)(v_2^2+v_3^2)}}}$$

Analysis and numerical result

If $v_1 = v_2 = v_3 = v$ then $e = 0$ which corresponds to a circular orbit (the points A,B and C are trivial to find). The result is also invariant to the change $v_1 \leftrightarrow v_3$ as they only have to be anti-parallel, it doesn't matter which one is A and which one is C.

If $v_2 = 0$ then $e = 1$, which corresponds to a parabola (implying points A and C should also coincide with $v_1 = v_3$). If $v_2 \neq 0$ label ratios $x \equiv v_1/v_2$ and $y \equiv v_3/v_2$ so

$$e = \sqrt{1 - \frac{4xy}{(1+x^2)(1+y^2)}}$$

By writing out the expression inside the square root

$$(1+x^2)(1+y^2) - 4xy = 1 + x^2y^2 + x^2 + y^2 - 4xy = (xy-1)^2 + (x-y)^2 \geq 0$$

it is evident the eccentricity can be calculated for any triplet (v_1, v_2, v_3) of numbers but is also bounded by $0 \leq e \leq 1$, as $x, y \geq 0$ (which also makes sense, if $e > 1$ the orbit would be hyperbolic and no two parallel velocities could exist if A and C don't coincide).

By substituting in the values $v_1 = 1\mathrm{km/s}$, $v_2 = 2\mathrm{km/s}$ and $v_3 = 3\mathrm{km/s}$ the following result is obtained:

$$\boxed{e = \sqrt{\frac{17}{65}}}\ (\approx 0.511408)$$