Physics Cup – TalTech 2025

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1 Problem 5: Satellite Orbit

A satellite orbits a planet. At point A, its speed is v_1. At point B, its speed is v_2 and its velocity vector forms a right angle with the velocity vector at point A. At point C, the velocity is exactly opposite to the velocity at point A, with a magnitude of v_3. Find the eccentricity of the orbit. Also, determine the exact numerical value of the eccentricity when $v_1 = 1$ km/s, $v_2 = 2$ km/s, and $v_3 = 3$ km/s.

2 Facts

In this section, I will enunciate some useful geometrical facts about an elliptical orbit. Some proofs are left in the appendix section. To start with, consider an ellipse Γ with major axis a and minor axis b and focal points F_1 and F_2. Set $c = \sqrt{a^2 - b^2}$ and O as the center of Γ.

Fact 1. *If P is a point on Γ and Q and R are points on the tangent to Γ at P such that $P \in \overline{QR}$, then:*

- $2a = F_1P + F_2P$
- $\angle F_1PQ = \angle F_2PR$

In simple terms: The total distance from any point on the ellipse to its two focal points adds up to $2a$. Also, if a ray comes from one focus, hits the tangent at the point P, it will bounce off at the same angle and head towards the other focus.

Fact 2. *Given a tangent line l_1 to Γ at P_1, there exist a unique different tangent line l_2 at P_2 such that $l_1 \parallel l_2$. In addition, $\overline{P_1P_2}$ has O as midpoint.*

Fact 3. *The geometrical locus of all points P such that its tangents to Γ are perpendicular, it's a circumference Ω of radius $R = \sqrt{a^2 + b^2}$ center at the center of Γ.*

3 Conservation laws

The first conservation law is energy conservation, from one focus:

$$E = -\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{1}$$

To find an expression for the angular momentum is trickier. Consider the minimum and maximum distances to the planet $d_1 = a - c$ and $d_2 = a + c$ with speeds u_1 and u_2 respectively. From here, the angular momentum density, h, is preserved, from which we can write the speed as $u = h/d$. Replacing this into the difference given by equation **(1)**, we arrive at:

$$h^2\left(\frac{1}{d_1^2} - \frac{1}{d_2^2}\right) = 2GM\left(\frac{1}{d_1} - \frac{1}{d_2}\right) \Rightarrow h^2 = 2GM\frac{d_1d_2}{d_1 + d_2}$$

Using $d_1d_2 = (a+c)(a-c) = a^2 - c^2 = b^2$ and **Fact 1**, it is clear that our conserved quantity is:

$$h^2 = GM\frac{b^2}{a} \tag{2}$$

4 Geometrical trick

By **Fact 2** and **Fact 3**, the points A, B, C, and D are constructed in such a way that their tangents intersect at points P, Q, P', and Q' as shown in **Figure 1**, forming the rectangle $PQP'Q'$. Let the angles formed by the tangent lines and the radial vectors from F_1 at A and B be ϕ and β, respectively. By using **Fact 1** and **Fact 2**, we can find the other angles shown in **Figure 1**.

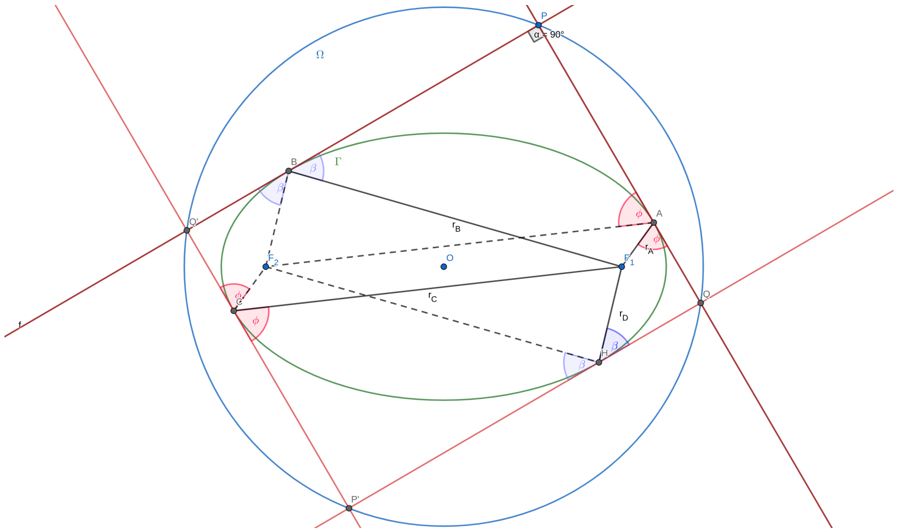

Figure 1: Geometrical construction

By using **Fact 2**, $\overline{CA}$ and $\overline{F_1F_2}$ have O as a midpoint, which makes CF_2AF_1 a parallelogram, from which the following relations come: [1]

$$r_A + r_C = r_B + r_D = 2a \tag{3}$$

From this figure, it is clear that the diameter can be found as:

$$(r_A \sin\phi + r_C \sin\phi)^2 + (r_B \sin\theta + r_D \sin\theta)^2 = 4R^2 = 4(a^2 + b^2)$$

[2] Using **(3)**

$$\sin\theta^2 + \sin\phi^2 = 1 + \left(\frac{b}{a}\right)^2 \tag{4}$$

5 Orbital Parameters

From equations **(3)** and the conservation of, $h = v_1 r_A \sin\phi = v_3 r_C \sin\phi$ it follows immediately that:

$$r_A = 2a\frac{v_3}{v_1 + v_3} \tag{5}$$

On the other hand, from energy conservation we can find the parameters GM as follows:

$$GM = \frac{v_1^2}{\left(\dfrac{2}{r_A} - \dfrac{1}{a}\right)} = \frac{v_1^2 a}{\left(\dfrac{v_1 + v_3}{v_3} - 1\right)} \Rightarrow GM = a v_1 v_3$$

Applying energy conservation to B,

$$v_2^2 = a v_1 v_3 \left(\frac{2}{r_B} - \frac{1}{a}\right)$$

$$r_B = 2a\frac{v_1 v_3}{v_2^2 + v_1 v_3} \tag{6}$$

[1]The same argument can be taken for B and D.
[2]**Fact 3** was used.

By using the equation **(2)** for the conservation of angular momentum density, $h = v_1 r_A \sin\theta = v_2 r_B \sin\phi$, and replacing **(5)** and **(6)** we arrive to:

$$\sin\theta = \left(\frac{v_2^2 + v_1 v_3}{v_2(v_1 + v_3)}\right)\sin\phi \tag{7}$$

Finally, the conservation of h^2 given by **(2)** and $GM = av_1v_3$ lead us to,

$$v_1 v_3 b^2 = (v_1 r_A \sin\phi)^2$$

from **(5)** we finally arrive to,

$$\sin\phi^2 = \frac{(v_1 + v_3)^2}{4v_1v_3}\left(\frac{b}{a}\right)^2 \tag{8}$$

6 Finding the eccentricity

We have all pieces set up to get $e^2 = 1 - b^2/a^2$. Therefore, replacing **(7)**, **(8)** into **(4)**,

$$2 - e^2 = \left(\frac{(v_1 + v_3)^2}{4v_1v_3}\right)\left[1 + \left(\frac{v_2^2 + v_1v_3}{v_2(v_1 + v_3)}\right)^2\right](1 - e^2)$$

For our sanity, the simplified expression is the following,

$$\boxed{e = \sqrt{\frac{v_2^2(v_1^2 + v_2^2 + v_3^2) + v_1^2v_3^2 - 4v_1v_3v_2^2}{v_2^2(v_1^2 + v_2^2 + v_3^2) + v_1^2v_3^2}}}$$

Replacing the particular case in which $v_1 = 1$ Km/s, $v_2 = 2$ Km/s, and $v_3 = 3$ Km/s, we obtain:

$$\boxed{e_p = \sqrt{\frac{17}{65}}}$$

Appendix: Proofs of the Facts

Assume for all the proofs that the point O is at the origin:

Fact 1: This is a well known fact regarding ellipses. For a complete proof, please look at it on the web.

Lemma 1. The tangent line at point $P(x_0, y_0)$ has a parametric equation:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

Proof. By deriving the ellipse equation, we can obtain:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

This gives the slope of the tangent at $P(x_0, y_0)$:

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -\frac{b^2 x_0}{a^2 y_0}.$$

Therefore, the equation of the tangent line in point-slope form is:

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0).$$

From where it follows immediately that:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

□

Fact 2:

Proof. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points on the ellipse where the tangents l_1 and l_2 meet. By **Lemma 1**, their tangent equations are:

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1 \quad \text{and} \quad \frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 1.$$

For l_1 and l_2 to be parallel, their slopes must be equal. Which means that,

$$-\frac{b^2 x_1}{a^2 y_1} = -\frac{b^2 x_2}{a^2 y_2} \quad \Longrightarrow \quad \frac{x_1}{y_1} = \frac{x_2}{y_2}.$$

The last equations mean that the tangents of its radial vector from the origin are equal, that is, fixing P_1 our other tangent point is of the form $(y_2, x_2) = k(x_1, y_1)$. These equations only satisfy the ellipse equation if $|k| = 1$. From this, the uniqueness of P_2 $(k = -1)$ follows immediately. In addition, it is trivial to see that for $k = -1$, O is the midpoint of $\overline{P_1 P_2}$.

□

Fact 3:
The proof is certainly not mine; you can find a derivation here. But for the sake of completeness, here it is:

Proof. The equation of the tangent at a point (u, v) on the ellipse is:

$$\frac{u}{a^2}x + \frac{v}{b^2}y = 1.$$

Solving for y, we obtain: [3]

$$y = -\frac{b^2 u}{a^2 v}x + \frac{b^2}{v}.$$

Using the abbreviations:

$$m = -\frac{b^2 u}{a^2 v}, \quad n = \frac{b^2}{v},$$

from the ellipse equation:

$$\frac{u^2}{a^2} = 1 - \frac{v^2}{b^2} = 1 - \frac{b^2}{n^2},$$

[3] Here, it's assumed the tangent is non-vertical.

we get:

$$m^2 = \frac{b^4 u^2}{a^4 v^2} = \frac{1}{a^2}\frac{b^4}{v^2}\frac{u^2}{a^2} = \frac{1}{a^2}n^2\left(1-\frac{b^2}{n^2}\right) = \frac{n^2-b^2}{a^2}.$$

Hence:

$$n = \pm\sqrt{m^2a^2+b^2} \Rightarrow y = mx \pm \sqrt{m^2a^2+b^2}.$$

Solving for (u,v), we obtain the parametric representation:

$$(u,v) = \left(-\frac{ma^2}{\pm\sqrt{m^2a^2+b^2}}, \frac{b^2}{\pm\sqrt{m^2a^2+b^2}}\right).$$

For a tangent passing through (x_0, y_0), we have:

$$y_0 = mx_0 \pm \sqrt{m^2a^2+b^2}.$$

Eliminating the square root leads to:

$$m^2 - \frac{2x_0y_0}{x_0^2-a^2}m + \frac{y_0^2-b^2}{x_0^2-a^2} = 0.$$

which has two solutions m_1, m_2 corresponding to the two tangents passing through (x_0, y_0). Hence, if the tangents meet at (x_0, y_0) orthogonally, the following equations hold:[4]

$$m_1m_2 = -1 = \frac{y_0^2-b^2}{x_0^2-a^2}.$$

Rearranging, we obtain:

$$x_0^2 + y_0^2 = a^2 + b^2.$$

Which describes a circle centered at the origin with radius $R = \sqrt{a^2+b^2}$. □

[4]Given a quadratic equation $x^2 + bx + c = 0$ with roots x_1 and x_2, it is a well-known fact that $x_1x_2 = c$.