Problem 5 - Satellite Orbits

Thrayambhakesh H
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§1 Problem

Problem

A satellite orbits a planet. At point A, its speed is v_1. At point B, its speed is v_2 and its velocity vector forms a right angle with the velocity vector at point A. At point C, the velocity is exactly opposite to the velocity at point A, with a magnitude v_3. Find the eccentricity of the orbit. Also determine the exact numerical value of the eccentricity when $v_1 = 1\text{km/s}$, $v_2 = 2\text{km/s}$, and $v_3 = 3\text{km/s}$.

§2 Solution

§2.1 Assumptions

We employ coordinate geometry to solve this problem. Assume the orbit to be an ellipse with the major axis along the axis represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Since the velocity at point C is exactly opposite to the velocity at point A, C is the mirror image of A with respect to the origin (center of conic).
Let the mass of the satellite be M_s, and that of the planet be M_p.
Without loss of generality, assume the planet P to be situated at the focus

$$P \equiv (-ae, 0)$$

The directrix corresponding to this focus is

$$x + \frac{a}{e} = 0$$

Similarly, assume that the coordinates of A, B, and C are

$$A \equiv (a\cos\theta_1, b\sin\theta_1) \;\; B \equiv (a\cos\theta_2, b\sin\theta_2) \;\; C \equiv (-a\cos\theta_1, -b\sin\theta_1)$$

§2.2 Calculations

Claim —

$$v_A^2 = \frac{GM_s}{a}\left(\frac{1 - e\cos\theta_1}{1 + e\cos\theta_1}\right)$$

Proof.

$$\text{Total energy of the satellite} = -\frac{GM_pM_s}{2a}$$

At point A, distance $PA = e \cdot$ (distance of A from the directrix).

$$PA = e\left(a\cos\theta_1 + \frac{a}{e}\right) = a(1 + e\cos\theta_1)$$

$$U_A = -\frac{GM_sM_s}{a(1 + e\cos\theta_1)} \implies v_A^2 = \frac{GM_s}{a}\left(\frac{1 - e\cos\theta_1}{1 + e\cos\theta_1}\right)$$

□

Similarly

$$v_B^2 = \frac{GM_s}{a}\left(\frac{1 - e\cos\theta_2}{1 + e\cos\theta_2}\right) \text{ and } v_C^2 = \frac{GM_s}{a}\left(\frac{1 + e\cos\theta_1}{1 - e\cos\theta_1}\right)$$

This implies

$$\frac{v_3}{v_1} = \frac{1 + e\cos\theta_1}{1 - e\cos\theta_1} \quad \text{and} \quad \cos\theta_1 = \frac{1}{e}\left(\frac{v_3 - v_1}{v_3 + v_1}\right)$$

Claim —

$$\tan\theta_1 \cdot \tan\theta_2 = 1 - e^2$$

Proof. The slope of the tangent to the ellipse at a point (x, y) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}\left(\frac{b^2}{a^2}\right)$$

Since the velocities at A and B are perpendicular to each other

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_B = -1$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_B = \left(-\frac{b}{a}\cot\theta_1\right)\left(-\frac{b}{a}\cot\theta_2\right) \implies \tan\theta_1 \cdot \tan\theta_2 = \frac{b^2}{a^2} = 1 - e^2$$

□

Now we try to relate v_1, v_2, v_3, e and $\cos\theta_2$

$$v_A^2 = \frac{GM_s}{a}\left(\frac{v_1}{v_3}\right)$$

$$\frac{v_B^2}{v_A^2} = \frac{v_2^2}{v_1^2} = \frac{v_3}{v_1}\left(\frac{1 - e\cos\theta_2}{1 + e\cos\theta_2}\right)$$

This implies

$$\cos\theta_2 = \frac{1}{e}\left(\frac{v_1v_3 - v_2^2}{v_1v_3 + v_2^2}\right)$$

Since

$$\tan^2\theta_1 \cdot \tan^2\theta_2 = \left(1 - e^2\right)^2$$

$$\left(e^2\left(\frac{v_1v_3 + v_2^2}{v_1v_3 - v_2^2}\right)^2 - 1\right) \cdot \left(e^2\left(\frac{v_1 + v_3}{v_3 - v_1}\right)^2 - 1\right) = \left(1 - e^2\right)^2$$

Let $p = \left(\frac{v_1 + v_3}{v_3 - v_1}\right)$ and $q = \left(\frac{v_1v_3 + v_2^2}{v_1v_3 - v_2^2}\right)$

This implies that

$$p^2q^2e^4 - (p^2 + q^2)e^2 + 1 = e^4 - 2e^2 + 1$$

giving us

$$e = \sqrt{\frac{p^2 + q^2 - 2}{p^2q^2 - 1}}$$

For the case of $v_1 = 1\text{km/s}$, $v_2 = 2\text{km/s}$, and $v_3 = 3\text{km/s}$

$$p^2 = 4 \text{ and } q^2 = 49 \implies e = \sqrt{\frac{17}{65}}$$