Physics Cup 2025 - Problem 1

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1 Outline of the solution

Initially, some non-null Lorentz force acts on the charges inside the metal ball as it starts to rotate which causes currents $\mathbf{j} = \sigma \mathbf{v} \times \mathbf{B} = \sigma(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}$ to flow in the ball's interior. Due to Joule's law, the currents come to a halt, and a time-independent charge distribution arises, in turn causing the fields produced by the rotating ball to reach a steady state. As the rotating ball is in a steady state and there are no driving electromotive forces, there shouldn't be any relative motion between the charges and the ball, i.e. for any charge the total force is zero:

$$\mathbf{F} = q\left(\mathbf{E}_{\text{in}} + (\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}\right) = \mathbf{0} \tag{1}$$

By solving Laplace's equation we will find the electrostatic potential *everywhere* in space, with which we can then easily calculate the generated electric field outside of the ball, denoted by $\mathbf{E}_{\text{out}}$. Once we find the field, we will simply use the given assumption $L \gg R$, treat the non-rotating ball as if it were in a uniform external electric field, and calculate its induced dipole moment. Finally, we calculate the interaction force by using:

$$\mathbf{F}_{\text{interaction}} = (\mathbf{p} \cdot \nabla)\,\mathbf{E}_{\text{out}} \tag{2}$$

2 Electric field inside of the rotating ball

Using equation (1) we find the electric field generated inside of the ball choosing the spherical coordinates and $\mathbf{B}$ and $\boldsymbol{\omega}$ to point in the $\hat{\mathbf{z}}$ direction:

$$\mathbf{E}_{\text{in}} = \mathbf{B} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\mathbf{B} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{B} \cdot \boldsymbol{\omega}) = -B\omega r(\sin^2(\theta)\,\hat{\mathbf{r}} + \sin(\theta)\cos(\theta)\,\hat{\boldsymbol{\theta}}) \tag{3}$$

Due to all fields being time-independent, i.e. $\nabla \times \mathbf{E}_{\text{in}} = \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$, we may express $\mathbf{E}_{\text{in}}$ as a gradient of a scalar potential field φ_{in}:

$$\frac{\partial \varphi_{\text{in}}}{\partial r} = B\omega r \sin^2(\theta) \qquad \frac{1}{r}\frac{\partial \varphi_{\text{in}}}{\partial \theta} = B\omega r \sin(\theta)\cos(\theta) \tag{4}$$

The following solution is easily verified: $\varphi_{\text{in}}(r,\,\theta) = \frac{1}{2}B\omega r^2 \sin^2(\theta) + C$ where C is a constant to be determined.

3 Electric field outside of the rotating ball

Let's now find the potential outside of the ball, denoted by φ_{out}. As there is no charge outside of the ball, the potential outside satisfies Laplace's equation: $\nabla^2 \varphi_{\text{out}} = 0$, i.e. we can expand the potential as a series in spherical harmonics which degenerate into Legendre polynomials with argument $\cos(\theta)$ due to axial symmetry about the z-axis. Keeping in mind that the potential must not diverge at infinity, we can write down its expansion:

$$\varphi_{\text{out}}\,(r,\,\theta) = \sum_{l=0}^{\infty} A_l r^{-1-l} P_l(\cos(\theta)) \tag{5}$$

The potential is a continuous function at the boundary $(r = R)$ of the ball, thus:

$$\sum_{l=0}^{\infty} A_l R^{-1-l} P_l(\cos(\theta)) = \frac{1}{2}B\omega R^2 \sin^2(\theta) + C = \frac{1}{2}B\omega R^2\left(1 - \cos^2(\theta)\right) + C \tag{6}$$

Evidently $\forall l \notin \{0, 2\}$, $A_l = 0$ (orthogonality of Legendre polynomials), along with $P_0(x) = 1$ and $P_2(x) = \frac{3x^2-1}{2}$, we are left with a much simpler equation:

$$\frac{A_0}{R} + \frac{A_2}{R^3}\frac{3\cos^2(\theta) - 1}{2} = \frac{1}{2}B\omega R^2\left(1 - \cos^2(\theta)\right) + C \tag{7}$$

This equation must be satisfied for all theta, which means that:

$$\begin{aligned} \frac{A_0}{R} - \frac{A_2}{2R^3} &= \frac{1}{2}B\omega R^2 + C \\ \frac{3A_2}{2R^3} &= -\frac{1}{2}B\omega R^2 \end{aligned} \tag{8}$$

The solutions for A_0 and A_2 in the system (8) are the following:

$$A_0 = CR + \frac{1}{3}B\omega R^3 \qquad A_2 = -\frac{1}{3}B\omega R^5 \tag{9}$$

Thus, the potential outside of the ball is given by:

$$\varphi_{\text{out}}(r,\, \theta) = \left(CR + \frac{1}{3}B\omega R^2\right)\frac{1}{r} - \frac{B\omega R^5}{3r^3}\frac{3\cos^2(\theta) - 1}{2} \tag{10}$$

The monopole term in the potential vanishes because the net charge of the ball is zero, i.e. $C + \frac{1}{3}B\omega R^2 = 0$, therefore the field outside is equivalent to that of a quadrupole:

$$\varphi_{\text{out}} = -\frac{1}{3}B\omega R^5\frac{3\cos^2(\theta) - 1}{2r^3} \tag{11}$$

Now, the field is simply $\mathbf{E}_{\text{out}} = -\nabla\varphi_{\text{out}}$:

$$\mathbf{E}_{\text{out}}(r,\, \theta) = -\frac{B\omega R^5}{r^4}\left(\frac{3\cos^2(\theta) - 1}{2}\hat{\mathbf{r}} + \sin(\theta)\cos(\theta)\hat{\boldsymbol{\theta}}\right) \tag{12}$$

On the z-axis, where the other ball is located ($\theta = 0$), the field is simply given by

$$\mathbf{E}_{\text{out}}(z) = -\frac{B\omega R^5}{z^4}\hat{\mathbf{z}} \tag{13}$$

4 Induced dipole moment of the other, non-rotating ball

Let's consider a metal ball of radius R in a uniform external electric field $\mathbf{E}_{\text{ext}}$. Due to the ball being conductive, a charge distribution on the surface will form such that the newly produced electric field inside cancels the external one, i.e. $\mathbf{E}_{\text{in}} = -\mathbf{E}_{\text{ext}}$. Let's consider a superposition of two charged spheres of the same radius R with charge densities ρ and $-\rho$ respectively, but whose centres are separated by a distance $\mathbf{x}$ such that a point inside the original ball at $\mathbf{r}$ is at distances $\mathbf{r}_+ = \mathbf{r} - \frac{\mathbf{x}}{2}$ and $\mathbf{r}_- = \mathbf{r} + \frac{\mathbf{x}}{2}$ away from the superposed spheres' centres. From Gauss' law:

$$4\pi\varepsilon_0 r_\pm^2 E_\pm = \pm\frac{4}{3}\pi r\pm^3\rho \quad\Longrightarrow\quad \mathbf{E}_\pm(r) = \pm\frac{\rho\mathbf{r}_\pm}{3\varepsilon_0} \tag{14}$$

Upon superposing the two spheres, we find that the electric field produced is uniform, and according to the uniqueness theorem, this superposition corresponds exactly to the original induced charge distribution on the ball:

$$\mathbf{E}_{\text{in}} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3}(\mathbf{r}_+ - \mathbf{r}_-) = \frac{\rho}{3\varepsilon_0}(\mathbf{R} - \frac{\mathbf{x}}{2} - \mathbf{R} - \frac{\mathbf{x}}{2}) = -\frac{\rho\mathbf{x}}{3\varepsilon_0} = -\mathbf{E}_{\text{ext}} \tag{15}$$

From the definition of polarisation $\mathbf{P} = \rho\mathbf{x}$, we conclude $\mathbf{P} = 3\varepsilon_0\mathbf{E}_{\text{ext}}$. The total dipole moment is therefore :

$$\mathbf{p} = \frac{4}{3}\pi R^3\mathbf{P} = 4\pi\varepsilon_0 R^3\mathbf{E}_{\text{ext}} \tag{16}$$

5 Interaction force between the balls

We derived the rotating ball's outside field and the stationary ball's induced dipole moment. Now we plug the values into the equation (2) and get the final answer for the modulus of the interaction force:

$$F(z) = p\frac{\partial E_{\text{out}}}{\partial z} = 4\pi\varepsilon_0 R^3\left(-\frac{B\omega R^5}{z^4}\right)\frac{\partial}{\partial z}\left(-\frac{B\omega R^5}{z^4}\right) = -16\pi\varepsilon_0 R^4 B^2\omega^2\left(\frac{R}{z}\right)^9$$

$$F = 16\pi\varepsilon_0 R^4 B^2\omega^2\left(\frac{R}{L}\right)^9 \tag{17}$$

Appendix

In our solution, we assumed the additional magnetic field created by the steady-state charge distribution to be negligible. We can calculate both the volume and surface charge density of the rotating ball. The volume charge density can be easily calculated using $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}_{\text{in}} = -2\varepsilon_0 B\omega$, while we find the surface charge density via:

$$\sigma(\theta) = -\varepsilon_0 \left(\frac{\partial \varphi_{\text{out}}}{\partial r} - \frac{\partial \varphi_{\text{in}}}{\partial r} \right) \Bigg|_{r=R} = \varepsilon_0 B \omega R \frac{3 - 5\cos^2(\theta)}{2}$$

Using these charge densities and their constant angular speed ω, we could calculate the magnetic field produced using the Biot-Savart law. However, we can immediately see that the $\mathbf{B}$ field would be proportional to $\mu_0 \varepsilon_0 \omega^2 R^2 = \frac{\omega^2 R^2}{c^2}$ which is completely negligible for all reasonable values of R and ω. We can conclude that our initial assumption was valid.

We should also discuss the other given strong inequality, $R \ll \sqrt{\rho/\mu\omega}$. The term on the right side of the inequality is identical to that of the skin depth of the conductor, with ω being the angular speed of the conductor instead of the regular alternating current frequency. As the skin depth is much greater than the characteristic size of the conductor, we can conclude that the magnetic field will indeed penetrate throughout the whole conductor and cause the eddy currents to flow at the beginning of the motion. The formation of eddy currents eventually leads to the described steady state.

Hereby we disregarded the attribution of the centripetal force that drives the charges' motion. Let's make a rough comparison between this centripetal force and the Lorentz force: $F_{\text{c}}/F_{\text{Lorentz}} \approx m_{\text{e}}\omega/eB \approx 10^{-11}\omega/B$. For any realistic ratio of ω/B, this ratio is tiny, i.e. a negligible fraction of the induced electric field is required to drive the circular motion of charges.

Let's derive equation (2). Consider a dipole moment $\mathbf{p}$ at position $\mathbf{r}$ in a non-uniform electric field $\mathbf{E}$. Fictitiously divide the dipole moment into 2 point charges $\pm q$ separated by a small distance $\boldsymbol{\delta}$, such that $\mathbf{p}$ is equal to $q\boldsymbol{\delta}$. Let's say that the negative charge is at position $\mathbf{r}_-$, then the positive charge is at position $\mathbf{r}_+ = \mathbf{r}_- + \boldsymbol{\delta}$. The force on the negative charge is simply $\mathbf{F}_- = -q\mathbf{E}(\mathbf{r}_-)$, while on the positive charge $\mathbf{F}_+ = q\mathbf{E}(\mathbf{r}_+) = q\mathbf{E}(\mathbf{r}_- + \boldsymbol{\delta})$. The force on the dipole is thus $\mathbf{F} = q(\mathbf{E}(\mathbf{r}_- + \boldsymbol{\delta}) - \mathbf{E}(\mathbf{r}_-)) = q(\mathbf{E}(\mathbf{r}_-) + \boldsymbol{\delta} \cdot \nabla\mathbf{E}(\mathbf{r}_-) - \mathbf{E}(\mathbf{r}_-)) = q\boldsymbol{\delta} \cdot \nabla\mathbf{E}(\mathbf{r}_-) = \mathbf{p} \cdot \nabla\mathbf{E}(\mathbf{r}_-)$. In the limit $\boldsymbol{\delta} \to \mathbf{0}$, $\mathbf{r}_- \to \mathbf{r}$, and we get $\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$.