2025 Physics Cup Problem 1

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1 Assumptions

1. Due to nonzero resistivity, no current will flow in the absence of external magnetic and electric fields.
2. The motion of the ball is non-relativistic.
3. The system is in a steady state.
4. The homogeneous magnetic field points in the same direction as the angular velocity (due to the B^2 in the final answer, this turns out to not matter).
5. The magnetic field generated by currents in the ball is negligible (justification for this one is given in the appendix).

2 Electric potential inside the ball

Note that there cannot be any radial currents or currents toward the poles, since otherwise by symmetry, charge would build up either at the center of the sphere or the poles, violating the steady-state assumption. Thus, the motion of any electron must be in circles around the rotation axis. Since the E and B fields are static in steady state, there cannot be a net electromotive force around any loop by Faraday's law, so there are no currents. Let an electron be a distance r^* from the rotation axis. We can conclude the following:

1. Since there are no currents, the particle is moving at a speed $v = \omega r^*$.
2. The net Lorentz force must be zero toward the axis of rotation, so $\omega r^* B + E = 0 \implies E = -\omega B r^*$, with the negative sign indicating an inward direction.

Integrating from the rotation axis, we get that the potential at a distance r^* inside the sphere is given by $\frac{1}{2}\omega B r^{*2} + V_0$ for some constant V_0.

3 Electric potential outside the ball

Let r bet the distance from the origin and θ the angle with the x-axis. At the boundary of the sphere, we can rewrite the electric potential as

$$\begin{aligned} V_0 + \frac{1}{2}\omega B R^2 \sin^2\theta &= V_0 - \frac{1}{2}\omega B R^2(\cos^2\theta - 1) \\ &= V_0 - \frac{1}{3}\omega B R^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \frac{1}{3}\omega B R^2 \\ &= \frac{3V_0 R + \omega B R^3}{3r} - \frac{\omega B R^5}{3r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right). \end{aligned}$$

This satisfies Laplace's equation (see appendix), so by the uniqueness theorem, this is the potential everywhere in free space outisde of the sphere. Since the sphere is neutral, the potential cannot fall off as $\frac{1}{r}$, so the first term must be zero, and we simply have

$$V(r,\theta) = -\frac{\omega B R^5}{3r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right),$$

for $r > R$.

4 Interaction force

Along the x-axis, $\theta = 0$, so

$$V(r) = -\frac{\omega B R^5}{3r^3} \implies E(r) = -\frac{\omega B R^5}{r^4}.$$

The interaction force will arise due to the polarization of the second ball, creating an electric dipole moment. Since $L \gg R$, we will assume a constant electric field of $E = -\frac{\omega B R^5}{L^4}$ for the purposes of calculating the polarization. The dipole moment of a conducting sphere in a uniform electric field is equal to

$$p = 4\pi\varepsilon_0 R^3 E = -\frac{4\pi\varepsilon_0 \omega B R^8}{L^4},$$

pointing inwards. A proof of this fact is given in the appendix. Finally, the interaction force is given by

$$p \cdot \nabla E = -\frac{4\pi\varepsilon_0 \omega B R^8}{L^4} \cdot \frac{4\omega B R^5}{L^5} = -\frac{16\pi\varepsilon_0 \omega^2 B^2 R^{13}}{L^9},$$

where the negative sign indicates an attractive force.

5 Appendix A: Justification for ignoring generated magnetic field

Let the total charge on the electrons in the ball be Q. The generated electric field is on the order of $\frac{Q}{R^2\varepsilon_0}$. The generated magnetic field is on the order of $\frac{\mu_0 Q\omega}{R}$. Thus, the ratio of the Lorentz force due to these fields on a charged particle is

$$\frac{\omega R B}{E} = \omega^2 R^2 \mu_0 \varepsilon_0 = \frac{\omega^2 R^2}{c^2} \ll 1,$$

since the motion of the ball was assumed to be non-relativistic.

6 Appendix B: Proof of potential formula

In spherical coordinates, the Laplacian is given by

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}.$$

For a cylindrically symmetric system, the last term is zero. As a check, we can calculate that for $V(r,\theta) = \frac{1}{r}$,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot -\frac{1}{r^2}\right) = 0.$$

For $V(r,\theta) = \frac{1}{r^3}(3\cos^2\theta - 1)$, we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-3}{r^2}(3\cos^2\theta - 1)\right) = \frac{6}{r^5}(3\cos^2\theta - 1)$$

and

$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(-6\cos\theta\sin^2\theta\cdot\frac{1}{r^3}\right)$$
$$= -\frac{6}{r^5\sin\theta}(2\sin\theta - 3\sin^3\theta)$$
$$= -\frac{6}{r^5}(2-3\sin^2\theta) = -\frac{6}{r^5}(3\cos^2\theta - 1),$$

so $\nabla^2 V = 0$ in this case as well. Hence, any linear combination of $\frac{1}{r}$ and $\frac{1}{r^3}(3\cos^2\theta - 1)$ will satisfy Laplace's equation, including the given formula for the potential. This ansatz is motivated by the multipole expansion for the potential using Legendre polynomials, given by

$$V(r,\theta) = \frac{C_1}{r} + \frac{C_2}{r^2}\cos\theta + \frac{C_3}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right).$$

7 Appendix C: Proof of polarization of sphere

Consider two homogeneous balls of charge $-q$ and $+q$ and radius R located at $(0,0)$ and $\mathbf{d} = (d,0)$, respectively, where $d \ll R$. The electric field at a point $\mathbf{r}$ is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q\mathbf{r}}{R^3} + \frac{q(\mathbf{r}-\mathbf{d})}{R^3}\right) = -\frac{q\mathbf{d}}{4\pi\varepsilon_0 R^3}.$$

Note that this configuration has no net charge in the intersection of the balls, and has a dipole moment of $q\mathbf{d} = -4\pi\varepsilon_0 R^3\mathbf{E}$. Hence, a conducting sphere in a uniform electric field $\mathbf{E}$ will set up surface charges equivalent to the intersection of two such homogeneous balls of charge, and develop a dipole moment of $\mathbf{p} = 4\pi\varepsilon_0 R^3\mathbf{E}$ to cancel out the E-field inside the sphere.