PROBLEM 1: INTERACTION FORCE

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Date: 13/14 October 2024

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Problem

The center of a metal ball with radius R is at the origin; the ball is in a homogeneous magnetic field B that is parallel to the x-axis. The ball rotates with an angular speed ω around the x-axis. At $x = L, y = z = 0$, there is another identical metal ball that does not rotate. Find the interaction force between the two balls, assuming that $L \gg R$ and $R \ll \sqrt{\frac{\rho}{\mu\omega}}$

1 Solution

1.1 General consideration

Here we just consider how should we solve the problem. Firstly B_0 creates a uniform magnetization and also it creates an uniform magnetic field inside the sphere. Also this magnetic field crates the nonzero charge density inside the ball. This ball then creats a magnetic field inside and outside the ball. This electrically and magnetically polarised ball crates a net field on the other ball, which crates a net force. There will be a force due to magnetization of the spheres. There will be force due to electrical interaction.

1.2 Approximation

Firstly, the problem states that the ball is nonmagnetic. This means that $\mu \approx \mu_0$, meaning that both balls are not polarised. First we exploit what does the second approximation in the problem means. We shall now estimate how large is the magnetic field inside the ball due to induced charge and how large is the field due to the bound currents. Magnetic field inside the ball put in the uniform magnetic field $\vec{B_0}$, will be:

$$\vec{B} = \vec{B_0} + \vec{B'} \tag{1}$$

Field $\vec{B'}$ of the charges moving in the ball, due to equation $\vec{J} = \frac{1}{\rho}(\vec{E} + \vec{v} \times \vec{B})$, will creats their own field. We now estimate this field:

$$B' \sim R\omega Q \frac{1}{R^2}\mu_0 \tag{2}$$

Charge that moves can be estimated trough the Ohm's as (Incucced voltage between center and the shell of the ball) :

$$Q\omega \frac{\rho}{R} \sim \omega R^2 B \tag{3}$$

Where B represents field in equation (1). Then we get:

$$B' \sim \frac{\omega \mu_0 R^4 B}{R^2 \rho} \tag{4}$$

If the ratio B'/B is denoted by K, than

$$K \sim \frac{\omega \mu_0 R^2}{\rho} \tag{5}$$

Now we recall from the problem that $R \ll \sqrt{\frac{\rho}{\mu\omega}}$, and combined with (5) we get:

$$K \ll \frac{\mu_0}{\mu} \tag{6}$$

Since the ball is non-magnetic:

$$K \ll 1 \tag{7}$$

This means, the field of the charges moving relative to the ball is negligible compared to the B_0. First approximation just tells us, that the second contribution to the first ball's polarisation, by the polarisation of the second ball is negliglible.
There is also one more consideration we need to take into the account. Will the magnetic field of the co-rotating charge significantly impact the magnetic field inside the ball? The answer is no, by the argument of the dimensional analysis, field of the rotating charge density will be:

$$B'' \sim \rho R^3 \omega \frac{\mu_0}{R}$$

$$B'' \sim \epsilon_0 (B_0 + B'' + B''') R^2 \omega^2 \mu_0$$

$$B'' \sim \frac{\omega^2 R^2}{c^2}(B_0 + B'' + B''')$$

Due to consideration of tensial strenght of most metals, we can safely assume that $R\omega \ll c$. Thus this contribution can be safely neglected. By same argument field of rotating surface charge will be:

$$B''' \sim \frac{\omega^2 R^2}{c^2}(B_0 + B'' + B''')$$

and as for charge density, this will be negligible compared to B_0.

1.3 Calculating the electric charge configuration

Charge inside the ball will obey the following equation:

$$\vec{J} = \frac{1}{\rho}(\vec{E} + \vec{v} \times \vec{B}) \tag{8}$$

A discussed before, $\vec{J}$ is small and can be set to zero. Than we get a simple equation:

$$\vec{E} = -\vec{v} \times \vec{B} \tag{9}$$

Now we can now approximate that the magnetic inside filed, created by the static charge relative to the ball is approximately zero (prove in section(1.2)) Thus the field inside the ball is uniform. now with the help of 9. We can get how the field should look in the interior of the ball:

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B}$$

$$\vec{E} = -\omega B \vec{s} \tag{10}$$

s represents the vector in the cylindrical coordinates, lying in the zy plane. Then the 1. Maxwell equation gives us:

$$\nabla \cdot \vec{E} = -2\omega B = \frac{\rho}{\epsilon_0}$$

$$\rho = -2\epsilon_0 B\omega \tag{11}$$

For σ one need to solve Laplace equation. Laplace for cylindrical coordinates is given by:

$$V = \sum_{l=0}^{\infty}(A_l R^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta) \tag{12}$$

P_l represents the l-th Lagedre's polynomial. Our boundary conditions are given by:

- V=0, when $r \to \infty$
- $V = V_c + \omega B \frac{R^2 \sin^2\theta}{2}$, when r=R

First condition gives us that all A_l must be 0. The second gives us

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_c + \omega B \frac{R^2 \sin^2\theta}{2} \tag{13}$$

Since (13) must hold for arbitrary θ, equation can only hold if:

$$B_l = 0; l \neq 1, 2$$

Then we obtain:

$$\frac{B_0}{R} + \frac{B_2}{2R^3}(2 - 3\sin^2\theta) = V_c + \frac{\omega B R^2}{2}\sin^2\theta \tag{14}$$

Since it must hold for arbitrary θ:

$$B_2 = -\frac{\omega B R^5}{3} \tag{15}$$

$$B_0 = V_c R + \frac{\omega B R^3}{3} \tag{16}$$

Thus potential outside the ball can be written as:

$$V = \frac{R}{r}(V_c + \frac{\omega B R^2}{3}) - \frac{\omega B R^5}{3r^3} P_2 \tag{17}$$

From $\vec{E} = -\nabla V$ we can obtain:

$$\vec{E} = \frac{R}{r^2} V_c \hat{r} + \frac{\omega B R^2}{3}(\frac{R}{r^2} - 3\frac{R^3}{r^4} P_2)\hat{r} - \frac{\omega B R^5 \sin\theta\cos\theta}{r^4}\hat{\theta} \tag{18}$$

Now the Inside radial electric field at the boundary is given by:

$$E_r = -\frac{2\omega Br}{3}(1 - P_2) \tag{19}$$

Surface charge density is given by equation $\frac{\sigma}{\epsilon_0} = E_{r,above} - E_{r,below}$. From (19):

$$\frac{\sigma}{\epsilon_0} = \frac{1}{R}V_c + \frac{\omega BR}{3}(3 - 5P_2) \tag{20}$$

By the conservation of charge, the total charge induced on the surface of the ball needs to be $\frac{8}{3}\pi R^3\epsilon_0 B\omega$. Thus we get:

$$\frac{8}{3}\pi R^3\epsilon_0 B\omega = \int_0^\pi (\frac{1}{R}V_c + \frac{\omega BR}{3}(3 - 5P_2))\epsilon_0 2\pi R^2 \sin\theta\, d\theta$$

$$\frac{4}{3}R^2 B\omega = \int_0^\pi (V_c + \frac{\omega BR^2}{3}(3 - 5P_2)\sin\theta\, d\theta = 2V_c + \frac{\omega BR^2}{3}6$$

$$V_c = -\frac{\omega BR^2}{3} \tag{21}$$

Thus charge density is:

$$\sigma = \frac{\epsilon_0\omega BR}{3}(2 - 5P_2) \tag{22}$$

Field that this charge distribution crates outside the ball is (in radial direction):

$$E_r = \frac{\omega BR^5 P_2(\cos\theta)}{r^4} \tag{23}$$

1.4 Electric force

Now electric field of this ball will polarise the second ball. Since the second ball is identical to the first one, this means that we can set $\rho \to \infty$ in equation (8). Thus we get:

$$\vec{E} = -\vec{v} \times \vec{B} \tag{24}$$

But since we have a static case and the ball does not rotate, the only way this equation can be true is by setting $\vec{E}_{in}$ to 0. Thus the second ball creates the polarisation(since we are working at the angle $\theta = 0$, $P_2 = 1$:

$$\vec{P} = -3\epsilon_0\vec{E} = 3\epsilon_0\frac{\omega BR^5}{L^4}\hat{x} \tag{25}$$

Force on the dipole can be calculated with the following equation:

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$$

So the force between the 2 balls is:

$$F = 3\epsilon_0\frac{\omega BR^5}{L^4}\frac{4\pi}{3}R^3\frac{\partial E}{\partial r}$$

$$F = 12\epsilon_0\frac{\omega BR^5}{L^4}\frac{8\pi}{3}R^3\frac{\omega BR^5}{L^5}$$

$$F = 16\pi\epsilon_0\frac{\omega^2 B_0^2 R^{13}}{L^9} \tag{26}$$

A Ball in uniform electrical field

If a ball with $\epsilon = \infty$ is put into the external uniform field E_0 it develops the polarisation. Neutral ball can be thought as a superposition of uniformly charges balls. After the external field is applied, charges in each of the charges ball will feel force. At the static state, the force, produced by the external field must be the same and opposite to the one, created by the polarisation. Field of uniformly charged ball inside the ball is $\vec{E} = \frac{\rho}{3}\vec{r}$. If the centers of the balls are apart by d, than:

$$\vec{E}_{in} = \frac{\rho}{3}\vec{d}$$

But since:

$$\vec{p} = Q\vec{d}$$

This leads to

$$\vec{E}_{in} = \frac{\rho}{3}\vec{d} = \frac{\rho V}{V3}\rho\vec{d} = \frac{\vec{P}}{3}$$

So

$$\vec{E_0} = -\frac{\vec{P}}{3} \tag{27}$$

B Force on the dipole

For an ideal dipole, the dipole moment is given by the equation $\vec{p} = \lim_{|\vec{d}|\to 0} q\vec{d}$. So the force on each charge is equal to $\pm E_i q_i$. Combined force is thus equal to

$$\vec{F} = q(\vec{E}_+ - \vec{E}_-)$$

. Since the point, at which this field is estimated are d apart, we rewrite this formula as:

$$\vec{F} = q\Delta\vec{E}$$

Delta of vector E can be rewritten by each component as:

$$\Delta E_i = (\nabla E_i) \cdot \vec{d}$$

Taking into the account all components:

$$\Delta\vec{E} = (\vec{d} \cdot \nabla)\vec{E}$$

So force can be rewritten as:

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E} \tag{28}$$

C Tensile strength

By dimensional analysis, $R\omega_{crit} = \sqrt{\frac{p_{crit}}{\rho}}$.

Material	Yield strength (MPa)	Ultimate tensile strength (MPa)	Density (g/cm³)
Steel, structural ASTM A36 steel	250	400–550	7.8
Steel, 1090	247	841	7.58
Chromium-vanadium steel AISI 6150	620	940	7.8
Steel, 2800 Maraging steel[4]	2,617	2,693	8.00
Steel, AerMet 340[5]	2,160	2,430	7.86
Steel, Sandvik Sanicro 36Mo logging cable precision wire[6]	1,758	2,070	8.00
Steel, AISI 4130, water quenched 855 °C (1,570 °F), 480 °C (900 °F) temper[7]	951	1,110	7.85
Steel, API 5L X65[8]	448	531	7.8
Steel, high strength alloy ASTM A514	690	760	7.8
Acrylic, clear cast sheet (PMMA)[9]	72	87[10]	1.16
Acrylonitrile butadiene styrene (ABS)[11]	43	43	0.9–1.53
High-density polyethylene (HDPE)	26–33	37	0.85
Polypropylene	12–43	19.7–80	0.91
Steel, stainless AISI 302[12]	275	620	7.86
Cast iron 4.5% C, ASTM A-48	130	200	7.3
"Liquidmetal" alloy[citation needed]	1,723	550–1,600	6.1
Beryllium[13] 99.9% Be	345	448	1.84
Aluminium alloy[14] 2014-T6	414	483	2.8
strength#Definition (unreinforced)[15]	55	55	

Figure 1: Critical tensions

This values all show that $(\omega R)_{critcal} \ll c$.