PhysCup25 Lu

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1 Setup

Since the balls are nonmagnetic there will be no magnetic interaction between the balls. However, Lorentz force acts on the charges in the rotating ball because they are in the presence of a magnetic field. The charges will rearrange until the Force from the electric field counteracts the Lorentz force, so there will be a electrostatic interaction between the balls.

Since $R \ll L$, the lowest order term of the multipole expansion is a good approximation for the electric fields of the balls. The electric field from the rotating ball will induce an electric dipole in the stationary ball. Since the dipole moment will be on the order of the electric field at L, the electric field at from the dipole on the rotating ball will be of even lower order than the electric field at L, so the electric field from external sources in the rotating ball will be approximately 0.

Thus, the force from the electric field of the rotating ball on the stationary ball should be a good approximation of the interaction force. Consequently, the induced dipole moment will be oriented such that the interaction force is attractive.

2 Justsification for $\vec{B} \approx B\hat{x}$ in the rotating ball

Because the charge density in the rotating ball is non-zero there will be a magnetic field from the movement of the charges due to it's rotation, and a magnetic field from the free currents in the ball. The following subsections will be dedicated to verifying that the ansatz where these effects are negligible (so $\vec{B} = B\hat{x}$ in the ball) is approximately the real solution for $\vec{B}$

The Lorentz force will be $q\vec{v}\times\vec{B} = q(\vec{\omega}\times\vec{r})\times\vec{B} = q\omega B(\hat{x}\times\vec{r})\times\hat{x} = q\omega B(\vec{r}-(\vec{r}\cdot\hat{x})\hat{x})$ which in cylindrical coordinates (s, θ, ϕ) is $q\omega Bs\hat{s}$. Since the force from the electric field must counteract the Lorentz force, $q\vec{E} = -q\omega Bs\hat{s} \Rightarrow \vec{E} = -\omega Bs\hat{s}$ from this formula, magnetic field contributions from currents and rotation can be calculated.

2.1 Magnetic field from rotation

Let B_r denote the magnetic field generated by rotation (at a arbitrary point). The electric field inside the rotating ball will be proportional to B and therefore so will the charge density. The charge density will be proportional to the current from rotation and so proportional to the magnetic field from rotation B_r(tl;dr $B_r \propto B$). The other relevant variables will be $\mu_0, \epsilon_0, \omega, R$ with the only dimensionless combination (up to exponent) being $\frac{\omega R}{\sqrt{\mu_0 \epsilon_0}} = \frac{\omega R}{c}$ where c is the speed of light. So $\frac{B_r}{B} = f(\frac{\omega R}{c})$ I will take the liberty to assume that the fastest point on the ball will be much slower than the speed of light, so the ball is rotating "slowly" so the magnetic field of the ball will be "small" or more precisely $\frac{B_r}{B}$ is small.

2.2 Magnetic field from currents

Let B_c denote the magnetic field generated by current (at an arbitrary point) $B_c \propto J \propto E \propto B$ where J is current density. Since B_c is also dependent on the quantities μ_0, ω, R, ρ: $\frac{B_c}{B} = f(\mu_0, \omega, R, \rho)$. So since the quantities μ_0, ω, R, ρ only have only 1 dimensionless combination(up to exponent):$\frac{\rho}{\omega \mu_0 R^2}$ which is $\gg 1$ as given in the problem. Since the resistivity is "large" it can be assumed that the current is "small" so, so the magnetic field from the current is "small". More precisely $\frac{B_c}{B}$ is small.

3 Interaction force calculation

The total dipole moment will have the same symmetries as the electric field.Since electric field is invariant under rotations about the x axis, and reflections about the yz plane at $x = 0$ the total dipole moment must be 0, the next term in the multipole expansion would the the quadrupole.

3.1 Finding the field from the rotating ball

Since a quadrupole potential is suspected possible quadrupole potentials will be investigated. Using a process analogous to dipole potentials are constructed, place a dipole of dipole moment $p\hat{x}$ at the origin and $-p\hat{x}$ at $(d, 0, 0)$ taking $lim_{d \to 0}$ of the potential. Since the potential associated with a dipole moment $\vec{p}$ is $\frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3}$, the potential of the quadrupole will be :

$$lim_{d \to 0} \frac{p}{4\pi\epsilon_0} \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{x-d}{((x-d)^2+y^2+z^2)^{\frac{3}{2}}} \right) = lim_{d \to 0} \frac{pd}{4\pi\epsilon_0} \frac{d}{dx} \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right).$$

Fix pd to a constant A, then the limit becomes:

$$\frac{A}{4\pi\epsilon_0} \frac{d}{dx} \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) = \frac{A}{4\pi\epsilon_0} \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{A}{4\pi\epsilon_0} \frac{-3x^2+r^2}{r^5}.$$

Notice (miraculously) that integrating the electric field inside the ball gives the potential to be $-\frac{1}{2}\omega B s^2 = -\frac{1}{2}\omega B(r^2 - x^2) + C_1$, so on the surface of the ball, $-\frac{1}{2}\omega B(R^2 - x^2) + C_1 = \frac{1}{2}\omega B x^2 + C_2$. So if $A = -\frac{2\pi\epsilon_0 \omega B R^5}{3}$ then the quadrapole potential is equal to the potential of the ball on the surface up to a additive constant. A point charge at the center has an equipotential surface on the surface of the ball, therefore the potential of the rotating ball on the surface can be expressed as the sum of the potentials of a point charge and the quadrupole constructed above. Since this satisfies the Dirichlet boundary condition, by the uniqueness theorem, the potential outside the rotating ball is the sum of the potential of a point charge and the quadrupole. Since the rotating ball has no net charge, the potential outside is simply the potential of the quadrupole constructed above: $\boxed{V_r = \frac{1}{6}\omega B R^5 \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}}$.

3.2 Induced dipole moment on the stationary ball

The electric field on the stationary ball is in the x direction, by rotational symmetry and is in an approximately uniform electric field $\vec{E} = E\hat{x}$. The potential from the ball on itself must then be, $E(x-L)$ up to an additive constant(for the same reasons as said in section 3.1 this constant can be ignored).the potential of a dipole with a dipole moment p at L is $\frac{p}{4\pi\epsilon_0}\frac{x-L}{r^3}$ so if $p = 4\pi\epsilon_0 R^3 E$ then the potential from the dipole moment is equal to the potential from the ball on it's surface. This constitutes a Dirichlet boundary condition and again, by uniqueness, the potential outside the ball is equal to the dipole potential, so it's dipole moment is $-4\pi\epsilon_0 R^3 \frac{dV_r}{dx}|_{(L,0,0)} = 4\pi\epsilon_0\omega B R^8 \frac{1}{L^4}$ so $\boxed{p = 4\pi\epsilon_0\omega B R^8 \frac{1}{L^4}}$

3.3 Force calculation

The magnitude of the force from a electric field on 2 point charges with charge q and $-q$ separated by a small distance in the x direction is $|q\vec{E}(x) - q\vec{E}(x + d)| \approx |\vec{E}(x) - q(\vec{E} + \frac{d\vec{E}}{dx}d)| = |qd\frac{d\vec{E}}{dx}| = |p\frac{d\vec{E}}{dx} = |p\frac{d^2V_r}{dx^2}|$, because by symmetry, $E_y = E_z = 0$ along the x axis. The interaction force is then a force of attraction with magnitude $\boxed{|F| = |\frac{16\pi\epsilon_0\omega^2 B^2 R^{13}}{L^9}|}$