Physics Cup 2025 Problem 1

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October 2024

1 Electric field of rotating sphere

The following argument is adapted from Macdonald (2002), but with a few simplifications and shortcuts.

We can assume that the nonrotating sphere has no effect on the charge distribution or currents in the rotating sphere, since $R \ll L$. Any effect of the nonrotating sphere on the charge distribution of the rotating sphere would be a higher order effect since the nonrotating sphere does not generate any electric or magnetic fields in a uniform magnetic field. We can hence analyse the steady state fields near the rotating sphere without considering the nonrotating sphere.

Using Ohm's Law,

$$\mathbf{J} = \frac{1}{\rho}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Where we have assumed drift velocity is much less than the tangential velocity due to rotation. Divergence of current density must be 0 at steady state, as any creation or removal of charge will result in a change in the electric field. Taking the curl of Ohm's law, we also find that the curl of the current density field is 0.

The eddy currents must satisfy

$$\nabla \cdot \mathbf{J} = 0, \quad \nabla \times \mathbf{J} = 0$$

It turns out that all nonzero fields that satisfy zero divergence and curl must decay slowly towards infinity. Since the current density outside the sphere must be exactly 0, the field in our case cannot possibly decay slowly. Intuitively, if we follow a field line, it must curl at some point in order to be contained within the sphere. This means that the only possible current field is where current everywhere is 0. The above is a special case from the argument given in the appendix of Nurge et al., 2017. The electric field has to cancel out the force by the magnetic field.

So, inside the sphere,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \implies \mathbf{E} = -\omega B \mathbf{r}_\perp, \text{where } \mathbf{r}_\perp = \begin{pmatrix} 0 \\ r_y \\ r_z \end{pmatrix}$$

Note that we have assumed that the tangential velocity due to rotation of the sphere is nonrelativistic, which allows us to ignore an "advection current" created by the motion of the material itself (as opposed to Ohmic current). Taking the divergence, we find that there is a constant volume charge density

$$\rho_e = -2\epsilon_0\omega B$$

Since $\mathbf{E} = -\nabla V$, inside the sphere,

$$\nabla V(r < R) = \omega B \mathbf{r}_\perp$$

Writing the gradient in terms of spherical coordinates (with θ measured from the x-axis)

$$\hat{\mathbf{r}}\frac{\partial V}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial V}{\partial \theta} = \omega B \mathbf{r}_\perp = \omega B r \sin(\theta)(\hat{\mathbf{r}}\sin\theta + \hat{\theta}\cos\theta)$$

$$\therefore \frac{\partial V}{\partial r} = \omega B r \sin^2\theta \implies V(r < R) = V_0 + \frac{1}{2}\omega B r^2 \sin^2\theta$$

Note that the potential is symmetric across the y-z plane through the center of the sphere, and is also azimuthally symmetric about the x axis. Even though the volume charge density is spherically symmetric, the potential is not spherically symmetric, due to surface charges.

Now we have to solve for the potential far away, outside the sphere. There is an exact solution outlined in Appendix 2, which is what Macdonald does in his paper. However, we can also find an estimate through some intuitive reasoning. Expanding the potential as a multipole, with a power series in r and arbitrary functions in θ,

$$V(r \gg R) = \sum_{l=1}^{\infty} \frac{1}{r^l} f_l(\theta)$$

Now we can consider each term individually, starting from the most significant term. Since we are only concerned with far-field effects, we only need to consider the largest nonzero term. Since the sphere is electrically neutral, there is no monopole moment, so there is no first order term. There will also be no second order term, as a dipole (which would be entirely created from surface charges) would lead to an unsymmetry in the internal electric field across the y-z plane. Hence there is only a third order quadrupole term. Intuitively, the quadrupole moment is nonzero as more positive charges buildup at the “equator” (y-z plane passing through the center), than at the poles. This would create a quadrupole moment.

Using dimensional analysis on the terms in the Laplacian (ω, B, and R), we find that the potential at $x = L$ is

$$V(r = L, \theta = 0) \propto \frac{\omega B R^5}{L^3} f_3(0) \sim -\frac{\omega B R^5}{L^3}$$

This is a unique solution for $V \propto r^{-3}$. We have assumed that the constant of proportionality (together with $f_3(0)$ which is the function in θ) is of order unity. The potential is negative as the higher density of positive charges are at the “equator”. This causes the electric field along the x-axis to point inwards

We can now calculate the electric field at the sphere at $x = L$, which allows us to determine the interaction force. We must also assume that the second sphere does not create any other eddy currents in the original sphere (which is valid since $R \ll L$), so

$$\mathbf{E}(r = L, \theta = 0) = -\nabla V \sim -\frac{\omega B R^5}{L^4}\hat{\mathbf{x}}$$

This turns out to be the exact answer with no constant of proportionality, even in the near field (See Appendix 2). The electric field can be treated as constant at the location of the nonrotating sphere, since $R \ll L$. Now we can find the surface charge distribution of the nonrotating sphere.

2 Electric force on nonrotating sphere

We can imagine the uniform field as being produced by two point charges. One at $x = L + b$ with charge $+q = 2\pi\epsilon_0 b^2 E$ and $x = L - b$ with charge $-q$ where $b \gg R$.

Using the method of image charges: A charge located distance b away from the center of a conducting sphere induces an electric field equal to that produced by an image charge at R^2/b from the center of the sphere (in the direction of the real charge), with charge $-qR/b$. We find that the electric field produced by the nonrotating sphere is equal to that produced by a dipole at the center of the sphere with dipole moment:

$$p = qd = 2q\frac{R^3}{b^2} = 4\pi\epsilon_0 R^3 E$$

Note that the dipole moment has the same direction as the E field. Now we claim that the force exerted on the sphere due to the (uniform) electric field is equal to the force exerted on a dipole in the same electric field.

The force exerted by the sphere on the two charges producing the electric field (which we will call "electric field charges") is equal and opposite to the force exerted on the sphere due to the electric field, by Newton's third law. Since the electric field created by the sphere is equal to that of a dipole with moment described above, the force exerted by the dipole on the electric field charges is equal to the force exerted by the sphere on the electric field charges. Then we use Newton's third law again: the force on the electric field charges due to the dipole is equal and opposite to the force on the dipole due to the electric field.

In other words: force on dipole due to E = - force on E due to dipole = - force on E due to sphere = force on sphere due to E. Hence, the force on the sphere is

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E} = 4\pi\epsilon_0 R^3 E \frac{4\omega B R^5}{L^5}\mathbf{x} = 16\pi\epsilon_0 \omega^2 B^2 \frac{R^{13}}{L^9}\hat{\mathbf{x}}$$

The interaction force is repulsive.

3 References

1. McDonald, K. T. (2002). Conducting sphere that rotates in a uniform magnetic field.

2. Nurge, M. A., Youngquist, R. C., Caracciolo, R. A., Peck, M., & Leve, F. A. (2017). A thick-walled sphere rotating in a uniform magnetic field: The next step to de-spin a space object. American Journal of Physics, 85(8), 596-610.

4 Appendix 1: Mysterious second strong inequality

It appears that we have not used the second strong inequality. It is somewhat extraneous, but we can use it to prove that even if there were eddy currents in the rotating sphere, the magnetic field would not be affected significantly.

The eddy current is at most of order:

$$J \sim \frac{vB}{\rho} \sim \frac{\omega R B}{\rho}$$

Which creates a magnetic field:

$$B' \sim \mu_0 J R \sim \frac{\mu_0 \omega B}{\rho} R^2$$

Compared to the applied magnetic field,

$$\frac{B'}{B} \sim \frac{\mu_0 \omega R^2}{\rho}$$

(Note that $\mu_0 = \mu$) The fraction is much less than one. Hence we can ignore any eddy currents created in the rotating sphere.

5 Appendix 2: Exact solution to the Laplacian

Now, to solve the Laplacian outside the sphere, we have to use Legendre polynomials to get an exact solution. The general solution for the potential outside the sphere is a superposition of the Legendre polynomials

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Since the potential must go to zero at infinity, $A_l = 0$. The potential is continuous at the boundary $r = R$, so we can equate the potential to the interior potential.

$$P_0 = 1, P_1 = \cos\theta, P_2 = (3\cos^2\theta - 1)/2$$

In the interior,

$$V(r < R) = \left(V_0 + \frac{\omega B r^2}{3} \right) P_0 - \frac{\omega B r^2}{3} P_2$$

Matching the Legendre polynomial outside the sphere at $r = R_+$ to the potential inside the sphere at $r = R_-$, we find that the only nonzero Legendre coefficients are:

$$B_0 = R V_0 + \frac{\omega B R^3}{3}, B_2 = -\frac{\omega B R^5}{3}$$

We can now find the potential at a distance $r = L$, and $\theta = 0$.

$$V(r = L, \theta = 0) = V_0 \frac{R}{L} + \frac{\omega B R^2}{3} \left(\frac{R}{L} - \frac{R^3}{L^3} \right)$$

It seems that it may be difficult to find V_0, but we can prove that the $1/L$ term will be 0 through a multipole argument. Since the rotating sphere is electrically neutral, it must have zero monopole moment far away. Hence, there cannot possibly be a monopole $1/L$ term! Hence we can conclude that $V_0 = -\omega B R^2/3$.

$$\therefore V(r = L, \theta = 0) = -\frac{\omega B}{3} \frac{R^5}{L^3}$$