Physics Cup 2025

Problem 1: Interaction Force

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Problem Statement

The center of a metal ball with radius R is at the origin; the ball is in a homogeneous magnetic field B that is parallel to the x-axis. The ball rotates with an angular speed ω around the x-axis. At $x = L, y = z = 0$ there is another identical metal ball that does not rotate. Find the interaction force between the two balls, assuming that $L >> R$ and $R << \sqrt{\rho/\mu\omega}$, where ρ denotes the ball's resistivity and μ is the permeability.

Dipole-Dipole Interaction

First, let us forget about the rotation and analyze what is happening. We know that a uniform magnetic field induces a magnetization on the sphere given by:[1]

$$\mathbf{M} = \frac{3}{\mu_0}\left(\frac{\mu_r - 1}{\mu_r + 2}\right)\mathbf{B} \tag{1}$$

So, both spheres have an induced dipole moment and they attract each other. It is a standard result that the force between two magnetized spheres is the same as that between two point dipoles.[2] The dipole moment of a sphere is given by

$$\mathbf{m} = \frac{4\pi}{3}R^3\mathbf{M} = \frac{4\pi R^3}{\mu_0}\left(\frac{\mu_r - 1}{\mu_r + 2}\right)B \tag{2}$$

which generates a magnetic field at the center of the other sphere that is given by

$$\begin{aligned}\mathbf{B}_{\text{sphere}} &= \frac{\mu_0}{4\pi}\left[\frac{3(\hat{x}(\hat{x}\cdot\mathbf{m}) - \mathbf{m}]}{L^3}\right] \\ &= \frac{\mu_0 m}{2\pi L^3}\hat{x}\end{aligned} \tag{3}$$

The interaction between a dipole and a magnetic field is given by $U = -(\mathbf{m}\cdot B)$ and so, the force would be given by

$$\begin{aligned}\mathbf{F}_{\text{dipole-dipole}} = -\nabla U = \nabla(\mathbf{m}\cdot\mathbf{B}_{\text{sphere}}) \\ = -\frac{3\mu_0 m^2}{2\pi L^4}\hat{x} \\ = -24\pi\frac{B^2}{\mu_0}\left(\frac{\mu_r - 1}{\mu_r + 2}\right)^2\frac{R^6}{L^4}\hat{x}\end{aligned} \tag{4}$$

The negative sign is just to represent that it is an attractive force.

Electric field due to Rotation

Now consider what happens when one of the spheres starts to rotate, the charges in the sphere cannot be moving with respect to the conductor at steady state as that would cause dissipation (due to the resistivity, remember that there is no emf in the sphere, the flux doesn't change). However, there is now a Lorentz force that acts on the electrons due to the magnetic field that is unaccounted for. This means that charges in the conductor must rearrange in a way so that the net electric field inside the conductor cancels the Lorentz force due to the magnetic field. This ensures that the electrons do not move with respect to the conductor and hence no dissipation (a steady state solution).

It is important to note that $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$ and not mv^2/r. This is because, if you think about a sphere rotating in vacuum with no external fields, the centripetal force on the electrons in the sphere is generated by internal forces in the sphere. This should still be the case here, and the electromagnetic forces do not contribute to the centripetal force and instead must cancel. Let us work in spherical coordinates,

$$\begin{aligned}\mathbf{E}(r < R) &= -\mathbf{v}\times\mathbf{B} \\ &= -(w\hat{x}\times\mathbf{r})\times B\hat{x} \\ &\Big[\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\Big] \\ &= -\omega B\mathbf{r} + \omega r B\cos\theta\mathbf{x} \\ &\Big[\hat{x} = \cos\theta\hat{r} - \sin\theta\hat{\theta}, x = r\cos\theta\Big] \\ &= -\omega Br\Big[\sin^2\theta\hat{r} + \sin\theta\cos\theta\hat{\theta}\Big]\end{aligned} \tag{5}$$

This is the electric field inside the sphere, let us calculate the corresponding charge density ρ. I used the same symbol as the resistivity, which is bad manners, but I believe that the resistivity does not matter in this case as there is no relative motion of electrons with respect to the conductors and so no dissipation (unless it plays some role in calculating carrier density??).

$$
\begin{aligned}
\rho &= \varepsilon_0 \nabla \cdot \mathbf{E} \\
&= \varepsilon_0 \left[\frac{1}{r^2} \frac{\partial \left(r^2 E_r \right)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(E_\theta \sin\theta \right) \right] \\
&= -\varepsilon_0 \omega B \left[3\sin^2\theta + 2\cos^2\theta - \sin^2\theta \right] \\
&= -2\varepsilon_0 \omega B
\end{aligned} \tag{6}
$$

This is quite a surprising result as it seems to suggest that the charge density inside the sphere is uniform, which is not obvious and I cannot think of an intuitive reason for why this might be the case. Ideally we would hope that the surface charge density is also uniform and the net charge on the sphere is zero, leading to no electric field outside the conductor. However, that might not be the case and we must rigorously find the surface charge density assuming it is not uniform.

From the symmetry of the system, clearly the surface charge density σ and the electrostatic potential φ can only depend on θ. Applying Poissons equation outside the sphere we have for $r > R$:

$$
\begin{aligned}
&\nabla^2 \varphi = 0 \\
&\implies \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \varphi}{\partial \theta} \right) = 0
\end{aligned} \tag{7}
$$

This can be solved in the same way we solve the Hydrogen atom, assume a separable solution of the form:

$$\varphi(r, \theta) = R(r)\Theta(\theta)$$

Substituting this into the differential equation:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) \Theta(\theta) + \frac{1}{r^2 \sin\theta} R(r) \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) = 0$$

Dividing by $R(r)\Theta(\theta)$ and multiplying through by r^2

$$\frac{1}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = -\frac{1}{\Theta(\theta) \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) = \lambda$$

This gives us two separate ordinary differential equations:

1. The radial equation:

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \lambda R(r) = 0$$

2. The angular equation:

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \lambda \Theta(\theta) = 0$$

The angular equation is solved by the associated Legendre polynomials $P_\ell(\cos\theta)$, with $\lambda = \ell(\ell+1)$. The radial equation with $\lambda = \ell(\ell+1)$ becomes:

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \ell(\ell+1) R(r) = 0$$

The general solution to this equation is:

$$R(r) = A r^\ell + B r^{-(\ell+1)}$$

Since, any solution can be written as a sum of separable solutions we have,

$$\varphi(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

However, as usual we assume that the potential vanishes at infinity, giving us a general solution,

$$\varphi = \sum_{\ell=0}^{\infty} \left(B_\ell r^{-(\ell+1)} \right) P_\ell(\cos\theta) \tag{8}$$

where $P_\ell(\cos\theta)$ are the Legendre polynomials and B_ℓ is determined by the boundary conditions. To get the boundary conditions, say the potential at the center of the sphere is φ_0, this is with respect to potential at infinity as always, then from the equation of the electric field (5) we have the potential just below the surface of the sphere

$$
\begin{aligned}
\varphi(r = R^-, \theta) &= \varphi_0 + \omega B \sin^2\theta \int_0^R r dr \\
&= \varphi_0 + \frac{1}{2} \omega B R^2 \sin^2\theta
\end{aligned} \tag{9}
$$

There are only zeroth and second order terms in this equation and so $B_\ell = 0$ if $\ell \neq 0, 2$. We have $\sin^2\theta = \frac{2}{3}(1 - P_2(\cos\theta))$, we have

$$
\begin{gathered}
\varphi(r = R^-, \theta) = \phi_0 + \frac{1}{3} \omega B R^2 - \frac{1}{3} \omega B R^2 P_2(\cos\theta)) \\
\text{applying boundary conditions at } r = R \\
B_0 = \phi_0 R + \frac{1}{3} \omega B R^3 \\
B_2 = -\frac{1}{3} \omega B R^5
\end{gathered} \tag{10}
$$

And this means that the electric field outside the sphere is actually non-zero, i.e the charge distribution on the surface must be non-uniform. We first calculate the electric field outside the sphere and once again apply boundary conditions on the surface to estimate the electric field.

$$\begin{aligned}\mathbf{E}(r>R) &= -\nabla\varphi \\ &= -\nabla\left(\frac{B_0}{r}+\frac{B_2}{2r^3}(3\cos^2\theta-1)\right) \\ &= \left(\frac{B_0}{r^2}+\frac{3B_2}{2r^4}(3\cos^2\theta-1)\right)\hat{r}+\frac{3B_2}{r^4}\sin\theta\cos\theta\hat{\theta}\end{aligned} \tag{11}$$

So the difference between the radial electric field at $r = R^+$ and $r = R^-$ would give us the surface charge density (which is non-uniform).

$$\begin{aligned}\sigma(\theta) &= \varepsilon_0(E(R^+)-E(R^-)) \\ &= \frac{\varepsilon_0\varphi_0}{R}+\frac{\varepsilon_0\omega BR}{6}(11-15\cos^2\theta)\end{aligned} \tag{12}$$

How to determine φ_0?? We are yet to use the fact that the metal ball was initially neutral, let us do that, charge on surface:

$$\begin{aligned}Q_{\text{surface}} &= 2\pi R^2\int_0^\pi \sigma(\theta)\sin\theta d\theta \\ &= 4\pi\varepsilon_0\varphi_0 R+\frac{\pi\varepsilon_0\omega BR^3}{3}(22-10) \\ &= 4\pi\varepsilon_0\varphi_0 R+4\pi\varepsilon_0\omega BR^3\end{aligned} \tag{13}$$

Charge inside would be

$$\begin{aligned}Q_{\text{inside}} &= \frac{4}{3}\pi R^3\rho \\ &= -\frac{8}{3}\pi\epsilon_0\omega BR^3\end{aligned} \tag{14}$$

But $Q_{\text{inside}} + Q_{\text{surface}} = 0 \implies \phi_0 = -\frac{B\omega R^2}{3}$ Which means $B_0 = 0$. Subbing this back into the electric field (11), we have the electric field outside the rotating sphere to be

$$\mathbf{E} = -\frac{\omega BR^5}{2r^4}(3\cos^2\theta-1)\hat{r}-\frac{\omega BR^5}{r^4}\sin\theta\cos\theta\hat{\theta} \tag{15}$$

Interaction due to Electric field

Great! So, the rotation of the first sphere creates an electric field in space given by (15), this electric field interacts with the second sphere and induces an electric dipole in it. This is quite a standard problem[1] if we assume that the electric field is nearly uniform over the second sphere and we can do that because $L >> R$. Electric field at the second sphere is given by ($\theta \sim 0$)

$$\mathbf{E} = -\frac{\omega BR^5}{L^4}\hat{x} \tag{16}$$

This induces a dipole moment of[1]

$$\mathbf{p} = 4\pi\varepsilon_0 R^3\mathbf{E} = -\frac{4\pi R^8}{L^4}\varepsilon_0\omega B\hat{x} \tag{17}$$

The force of interaction of this dipole with the electric would be given by

$$\begin{aligned}\mathbf{F}_{\text{electric}} &= \nabla_{\theta=0,r=L}(\mathbf{p}\cdot\mathbf{E}) \\ &= \nabla_{\theta=0,r=L}(pE_r) \quad \left(\text{as } \hat{\theta}\perp\mathbf{p}\right) \\ &= p\nabla_{\theta=0,r=L}\left(\frac{\omega BR^5}{2r^4}(3\cos^2\theta-1)\right) \\ &= -p\frac{4\omega BR^5}{L^5}\hat{x} \\ &= -\frac{4\pi R^8}{L^4}\varepsilon_0\omega B\frac{4\omega BR^5}{L^5}\hat{x} \\ &= -16\pi\varepsilon_0\omega^2B^2\frac{R^{13}}{L^9}\hat{x}\end{aligned} \tag{18}$$

The expression is quite weird, but the units work out. This is an attractive force. It is important to understand that though we treated the sphere as a point dipole, since the electric field was not actually uniform and had a gradient, it led to a force. A uniform electric field would not have lead to a force, this is why it was important to take the derivative in general and compute it at $\theta = 0$ and $r = L$, as in theory there might have been even a tangential component (not possible due to the symmetry of the problem of course). Note that any force on the second sphere must necessarily be due to the first sphere as, the magnetic field is uniform and therefore exerts no force on the second sphere.

Corrections to the Magnetic field

I believe that there is another term missing in the equation due to the rotation, since the sphere has a non-uniform charge distribution and rotates, there could be an additional magnetic field generated outside the sphere and this would further contribute to the interaction (through the dipole moment). Let us use superposition to evaluate this magnetic field, first consider the magnetic field, purely due to the charge inside the ball.

The volume element is described in FIG. 1a, it has a volume $dV = 2\pi r dr dz$ and would carry a current $I = \rho\omega dV$. Okay, here I will start abusing the assumption $R << L$, ideally I should do to some arbitrary order and remove all terms before it, but since there are elements of order $(R/L)^9$ in the previous section, I will be quite sloppy. Specifically, I will not only assume that the

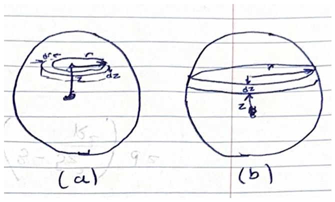

FIG. 1: Volume and Surface Elements used, z is along the direction of x, sorry for confusing notation.

magnetic field due to any such loop in FIG. 1a would be constant over the second sphere but also that every loop is at distance L from the second sphere (even though they are actually at distance $L - z$). Okay, so magnetic field due to such a loop in FIG. 1a at the other sphere would be $B = \mu_0 I r^2/2(r^2 + L^2)^{3/2} \approx \mu_0 I r^2/2L^3$. Ideally one would expand to larger order in (r/L) and compare, but my hope is to get a non-zero contribution of this lower order term so that I can claim that any higher order terms can be neglected. Basically, I am trying to use $R << L$ to get the lowest order contribution to each effect, I feel is relevant even though the orders might not be consistent between effects, but at least I wouldn't be missing any physics. The total magnetic field due to all such loops:

$$\begin{aligned}\mathbf{B}_{\text{bulk}} &= \frac{\pi\mu_0\rho\omega}{L^3}\int_{-R}^{R} dz \int_0^{\sqrt{R^2-z^2}} r^3 dr \\ &= \frac{4\pi}{15}\mu_0\rho\omega\frac{R^5}{L^3} \\ &= -\frac{8\pi}{15}\varepsilon_0\mu_0\omega^2 B\frac{R^5}{L^3}\hat{x}\end{aligned} \tag{19}$$

Now, let us calculate the contribution due to the surface charge, the element is described in FIG. 1b. $\cos\theta = z/R$ and so the surface charge density $\sigma(\theta)$ can be written as

$$\begin{aligned}\sigma(z) &= -\varepsilon_0\frac{B\omega R}{3} + \varepsilon_0\frac{B\omega R}{6}(11 - 15z^2/R^2) \\ &= \frac{\varepsilon_0 B\omega R}{2}\left(3 - 5\frac{z^2}{R^2}\right)\end{aligned} \tag{20}$$

The charge of this surface element would be $\sigma(z)2\pi r dz$, where $r = \sqrt{R^2 - z^2}$. As before, the magnetic field at the second sphere would be given by

$$\begin{aligned}B_{\text{surface}} &= \frac{\mu_0}{2L^3}\int_{-R}^{R} dz 2\pi\omega r^3\sigma(z) \\ &= \frac{\varepsilon_0 B\omega^2 R}{2}\frac{\mu_0\pi}{L^3}\int_{-R}^{R} dz(R^2 - z^2)^{3/2}\left(3 - 5\frac{z^2}{R^2}\right) \\ &= \frac{\varepsilon_0 B\omega^2 R}{2}\frac{\mu_0\pi}{L^3}\frac{13\pi}{6}R^4 \\ &= \frac{13\pi^2}{12}\varepsilon_0\mu_0\omega^2 B\frac{R^5}{L^3}\hat{x}\end{aligned} \tag{21}$$

Clearly, the surface and bulk magnetic fields are unequal therefore the net extra magnetic field generated due to rotation is given by

$$\begin{aligned}\mathbf{B}_{\text{correction}} &= \pi\varepsilon_0\mu_0\omega^2 B\frac{R^5}{L^3}\left(\frac{13\pi}{12} - \frac{8}{15}\right) \\ &= \pi\varepsilon_0\mu_0\omega^2 B\frac{R^5}{L^3}\frac{65\pi - 32}{60}\hat{x}\end{aligned} \tag{22}$$

This magnetic field once again would act as a force on the second sphere's magnetic moment, given by

$$\begin{aligned}\mathbf{F}_{\text{correction}} &= \nabla(\mathbf{m}\cdot\mathbf{B}_{\text{correction}}) \\ &= -m\pi\varepsilon_0\mu_0\omega^2 B\frac{R^5}{L^4}\frac{65\pi - 32}{20}\hat{x}\end{aligned} \tag{23}$$

Actually, this non-uniform field would generate a non-uniform magnetization on the second sphere, but all these are higher order contributions and I will ignore them and just use the dipole moment from before

$$\mathbf{F}_{\text{corr.}} = -\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\left(13\pi - \frac{32}{5}\right)\pi^2\varepsilon_0\omega^2 B^2\frac{R^8}{L^4}\hat{x} \tag{24}$$

Combining everything together we have

$$\boxed{\begin{aligned}\mathbf{F}_{\text{total}} &= \mathbf{F}_{\text{dipole-dipole}} + \mathbf{F}_{\text{electric}} + \mathbf{F}_{\text{correction}} \\ &= -\left(24\pi\frac{B^2}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)^2\frac{R^6}{L^4} + 16\pi\varepsilon_0\omega^2 B^2\frac{R^{13}}{L^9} + \left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\left(13\pi - \frac{32}{5}\right)\pi^2\varepsilon_0\omega^2 B^2\frac{R^8}{L^4}\right)\hat{x}\end{aligned}} \tag{25}$$

Since, $\mu_r = 1$

$$F = -16\pi\varepsilon_0\omega^2 B^2\frac{R^{13}}{L^9}\hat{x} \tag{26}$$

Attractive force

[1] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, New York, 1998).

[2] B. F. Edwards, D. M. Riffe, J.-Y. Ji, and W. A. Booth, Interactions between uniformly magnetized spheres, American Journal of Physics **85**, 130 (2017), https://pubs.aip.org/aapt/ajp/article-pdf/85/2/130/13113803/130_1_online.pdf.