Problem No 1 - Interaction force

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Let us first note, that the interaction force can only have a component, directed along x-axis. Components in y and z direction would cancel due to the symmetry. Also since both balls are metal, charge can move inside of them, as long as the whole ball remains neutral. Since the second ball is not rotating, magnetic field at that point has no effect on the ball, therefore only electric field, which can due to the fact, that $L >> x$, be assumed to be roughly constant over the ball, can produce a net force on second ball.

Firstly, let's consider only the rotating ball in uniform magnetic field, $\vec{B} = B\hat{x}$. On charges inside metal there acts magnetic force per unit charge $\vec{f} = \vec{v} \times \vec{B} = (\vec{\omega} \times \vec{r}) \times \vec{B}$, which is always directed away from the axis of rotation(x-axis) anywhere in the ball. Thus, positive charge is being pushed outward and negative inward, resulting in so-called eddy currents forming in the ball(figure 1). This current density is given by the Ohm's law,

$$\vec{j} = \frac{1}{\rho}(\vec{E} + \vec{v} \times \vec{B})$$

where $\vec{j}$ is the current density and ρ ball's specific resistivity.

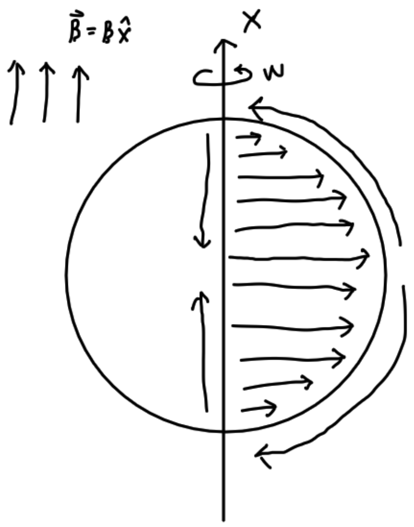

Figure 1: Eddy currents in rotating ball

In given problem, this, at the beginning, equals $\frac{1}{\rho}(\vec{v}\times\vec{B}) = \frac{1}{\rho}r\omega\sin\theta B\hat{s}$ (directed away from x-axis) This means that charge will be relocated, until $\nabla \cdot \vec{j} = 0$, when stationary state will be reached. Then there should be induced electric or/and magnetic field, that would nullify the effect of external field B. Because $\vec{f}$ is directed outward and strongest around equator, we can expect, that positive charge will be gathered around that area, while negative charge will be piled near the center(figure 2). For this our initial guess could be, that to outside points, metal ball could act as an electric quadrupole.

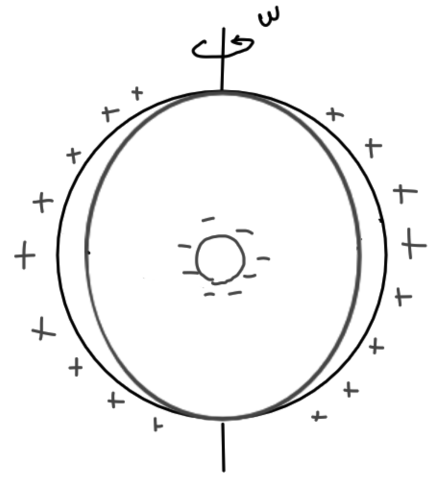

Figure 2: Charge distribution in a rotating ball

We can easily see that $\nabla \cdot \vec{j} = 0$ would be satisfied, when $\vec{E}$ would equal $-\vec{v} \times \vec{B}$, so when

$$\vec{E} = -r \sin\theta \omega B \hat{s}$$

where $\hat{s}$ is unit vector directed away from x-axis.

Then calculating potential on the surface of sphere, $V = -\int_O^{(R,\theta)} \vec{E} \cdot d\vec{l}$, while setting $V = 0$ at $r \longrightarrow \infty$ and then denoting with V_0 potential at the center of a sphere, so at the origin, by tracking the integral first along the x-axis and then directly outward(figure 3), we obtain

$$V(R,\theta) = V_0 - \int_0^{R\sin\theta} -\omega B s \, ds$$

$$V(R,\theta) = V_0 + \frac{1}{2}\sin^2\theta R^2 \omega B$$

We want to know, what is the potential for points outside of the ball - due to the uniqueness theorem it is determined by potential on the sphere with radius R and the fact, that V vanishes at large distances. Here we won't consider any secondary effects of second ball, as their contribution is neglectable. In order to determine potential in outer region, we need to solve Laplace's equation,

$$\nabla^2 V = 0$$

because there is no free charge, again neglecting the second metal ball.

General solution to Laplace's equation in spherical coordinates is

$$V = \sum_{l=0}^{\infty} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(\cos\theta)$$

where $P_l(x)$ are Legendre polynomials and A_l and B_l constant terms specific for each l. *Legendre polynomials are orthogonal functions, therefore $\int_{-1}^{1} P_m(x)P_n(x)\,dx \neq 0$ only if $m = m$, which allows determining values of coefficients with known potential at boundaries using Fourier's trick.*

With requirement that potential goes to zero at large r, A_l must be zero for all l. Also we know potential at $r = R$, so

$$V_0 + \frac{1}{2}\sin^2\theta R^2 \omega B = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

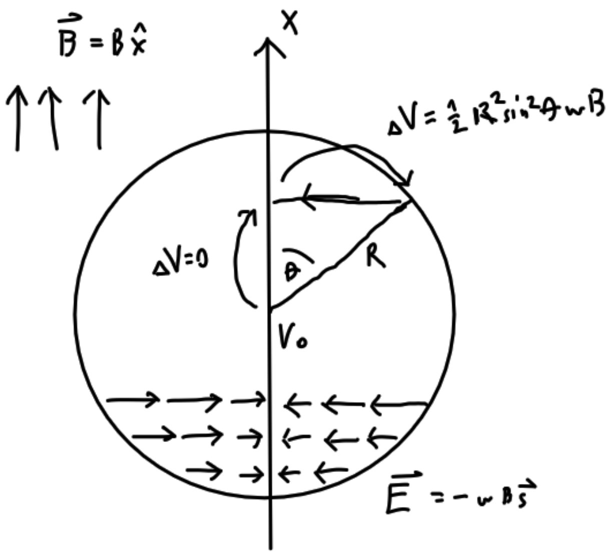

Figure 3: Potential on surface of the ball

We can rewrite the polar angle dependance on the left in terms of Legendre polynomials,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{3}(2P_2(\cos\theta) + 1))$$

$$\sin^2\theta = \frac{2}{3}P_0(\cos\theta) - \frac{2}{3}P_2(\cos\theta)$$

because $P_0(x) = 1$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

So the boundary requirement is then

$$V_0 + \frac{1}{2}R^2\omega B(\frac{2}{3}P_0(\cos\theta) - \frac{2}{3}P_2(\cos\theta)) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

due to the orthogonality all the terms in the sum on the right, that don't include P_0 or P_2, can't have any influence, thus their B_l is zero. We are left with

$$(V_0 + \frac{1}{3}R^2\omega B)P_0(\cos\theta) - \frac{1}{3}R^2\omega B P_2(\cos\theta) = \frac{B_0}{R^1}P_0(\cos\theta) + \frac{B_2}{R^3}P_2(\cos\theta)$$

from here we obtain

$$B_0 = R(V_0 + \frac{1}{3}R^2\omega B)$$

$$B_2 = -\frac{1}{3}R^5\omega B$$

and potential for point $(r, \cos\theta)$ is then

$$V = \frac{R(V_0 + R^2\omega B)}{r} - \frac{R^5\omega B}{6r^3}(3\cos^2\theta - 1)$$

The first term corresponds to potential due to the single charge in the origin and second to the potential of perfect electric quadrupole at the origin, however due to the fact, that metal ball is neutral, there should be no fist term in potential multipole expansion, thus

$$V_0 = -R^2\omega B$$

the ball acts as a electric quadrupole at the origin, with quadrupole electric moment

$$\vec{Q} = -\frac{1}{6}R^5\omega B 8\pi\epsilon_0\hat{x} = -\frac{4}{3}\pi\epsilon_0 R^5\omega B\hat{x}$$

Then electrical field, along x-axis, equals

$$\vec{E_x} = -\frac{\partial}{\partial x}V(r = x, \theta = 0)$$

$$\vec{E_x} = -\frac{R^5\omega B}{x^4}\hat{x}$$

At $x = L$ this field equals $\vec{E_L} = -\frac{R^5\omega B}{L^4}\hat{x}$ and is constant over the entire second ball. Therefore we must investigate, what happens to the metal ball in uniform electric field.

The charges in the metal ball, that is placed in the uniform electrical field, are free to move and are pushed along the field lines, until stationary state is reached. Then, according to Ohm's law, electric field induced by dislocated charges should cancel external field, $\vec{E_i} = -\vec{E}$, so induced field should also be constant inside the ball.

This condition is met, where there are two identical balls, one with charge $+q$ and the other with $-q$ evenly distributed over balls, displaced by tiny distance $\vec{s}$(figure 4)). Electrical field inside one ball is

$$\vec{E} = \frac{q\vec{r}}{4\pi\epsilon_0 R^3}$$

therefore field of negative ball at origin and positive at $\vec{s}$ is

$$\vec{E} = \frac{q(\vec{r} - \vec{s})}{4\pi\epsilon_0 R^3} + \frac{-q\vec{r}}{4\pi\epsilon_0 R^3}$$

$$\vec{E} = -\frac{q\vec{s}}{4\pi\epsilon_0 R^3}$$

$$\vec{E} = -\frac{\vec{p}}{4\pi\epsilon_0 R^3}$$

which is constant. When $s \longrightarrow 0$ points with $r > R$ perceive balls as point charges in their centers, which are displaced by $\vec{s}$ - electric field for exterior points is the same as the electric field of dipole moment $\vec{p}$ at origin,

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

If the interior field should be canceled,

$$\vec{E} - \frac{\vec{p}}{4\pi\epsilon_0 R^3} = 0$$

$$\vec{p} = 4\pi\epsilon_0 R^3\vec{E}$$

For the "outside" metal ball then acts as electric dipole with this dipole moment. If we set $\vec{E}$ to equal $\vec{E_L}$, we get

$$\vec{p} = -\frac{4\pi\epsilon_0 R^8\omega B}{L^4}\hat{x}$$

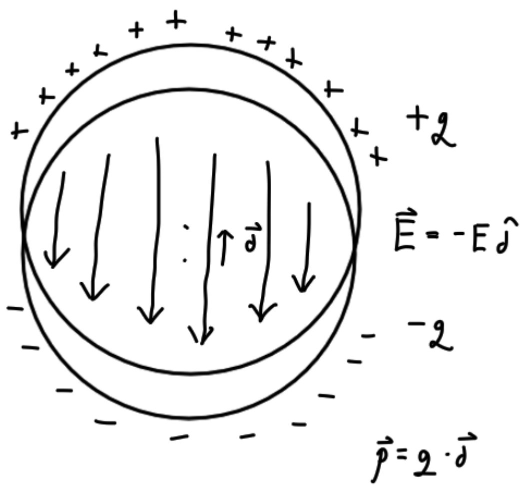

Figure 4: Metal ball in uniform external field

Force, acting on a dipole $\vec{p}$ in a electric field $\vec{E}$ is

$$\vec{F} = (\vec{p}\nabla\cdot)\vec{E}$$

since we only have component of dipole along x-axis, this simplifies to

$$\vec{F_x} = \frac{4\pi\epsilon_0 R^8 \omega B}{L^4} R^5 \omega B \frac{\partial}{\partial x}\frac{1}{x^4}\hat{x}$$

$$\vec{F_x} = -\frac{16\pi\epsilon_0 R^{13}\omega^2 B^2}{L^4 x^5}\hat{x}$$

and finally, the interaction force between balls is

$$\vec{F} = -\frac{16\pi\epsilon_0 R^{13}\omega^2 B^2}{L^9}\hat{x}$$

We can easily check, that the dimensions are indeed correct and also force is dependant on quantities as one would expect.