Physics Cup 2025 Problem 1

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1 Summary

The induced electric field in the rotating sphere is first found using the charge conservation equation and cylindrical symmetry. Next, due to the symmetry of the setup we hypothesise that the total field outside the rotating sphere is given by a linear quadrupole field. This field is then shown to satisfy the required boundary conditions - by uniqueness, this is the only valid field. Finally, the method of images is used to evaluate the force between the two spheres.

2 Assumptions

1. The first assumption $L >> R$ tells us that the charge distributions on one sphere will not affect the charge distributions on the other too much. This suggests that we can ignore next-to-leading order image charges etc. Basically, just the leading order term in the interaction force is sufficient.

2. The second assumption $R << \sqrt{\frac{\rho}{\mu\omega}}$ allows us to neglect self-induction effects in the sphere. When the magnetic field is turned on, there will initially be some changing electromagnetic fields within the sphere, with timescale $\tau \sim \frac{1}{\omega}$. However, thanks to the assumption, we can safely assume that in the steady state, all fields permeate the sphere at the given values without attenuation.

   Another way to interpret this assumption would be that, since the ball is metal and hence has low resistivity ($\rho \sim 10^{-8}$ Ωm for most metals), ω must be very low i.e. $\omega R << c$. Not only does this mean we get to avoid thinking about relativstic effects, it also means that the magnetic field inside the sphere will not significantly differ from B as a result of the induced charge distribution, as we will discuss at the end of Section 4.

3. While we are told that the magnetic field is parallel to the x-axis, it is not explicitly stated whether the angular velocity vector $\vec{\omega}$ is oriented parallel or antiparallel to the x-axis. We will assume in this solution that $\vec{\omega} = \omega\hat{x}$ (where $\omega > 0$), but we will show that the direction of $\vec{\omega}$ does not matter

for this problem (i.e. the interaction force will depend on an even power of ω).

4. We will solve for a general case where the permittivity of the conducting sphere is ϵ. We will show that the value of ϵ will not matter when computing the interaction force.

3 Inside the Rotating Sphere

Figure 1 shows the physical situation. Initially, when the magnetic field is turned on, the electrons in the rotating sphere experience a Lorentz force $-e(\vec{v} \times \vec{B})$. e is the elementary charge $1.602 \cdot 10^{-19}$ C, $\vec{v}$ is the charge velocity and $\vec{r}$ is the position vector from the origin. When the sphere eventually reaches electrostatic equilibrium, there will be some electric field $\vec{E}_{in}$ induced inside the sphere, leading to a charge density $\rho_{in} = \epsilon \nabla \cdot \vec{E}_{in}$ inside the sphere and some surface charge density outside. Since the system is symmetric about the x-axis, the resulting charge distribution must also have this symmetry. Hence, it will be good to analyze the fields and charge distributions by taking a cross-section of the sphere as shown in Figure 1, with the x-axis pointing out of the page. The cylindrical coordinates (r, θ, x) will be used to analyze the fields and charge distributions.

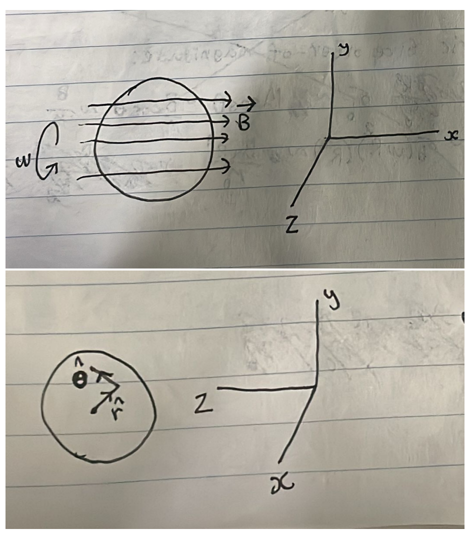

Figure 1: Top: The physical situation, with the axes and direction of rotation specified. Bottom: Circular cross section with the x-axis pointing out of the page; cylindrical coordinates (r, θ, x) are used.

The current density is $\vec{J} = \frac{1}{\rho}(\vec{E}_{in} + \vec{v} \times \vec{B}) = \frac{1}{\rho}(\vec{E}_{in} + \omega r B \ \hat{r})$ since the charges cannot move relative to the conductor by the definition of electrostatic equilibrium. In the steady state, the charge conservation equation reads

$$\nabla \cdot \vec{J} = \frac{1}{r}\frac{\partial (rJ_r)}{\partial r} + \frac{1}{r}\frac{\partial J_\theta}{\partial \theta} + \frac{\partial J_x}{\partial x} = 0 \tag{1}$$

Due to cylindrical symmetry, the components of $\vec{J}$ cannot depend on θ, hence the second term vanishes. Because of this, we require $J_r = 0$ everywhere inside the sphere. To see this, imagine we have a nonzero J_r at a point (r, θ, x). Then it is impossible for said current to form a closed loop and flow back to the original point because for all points with the same r and x, the current flow will be radially outward or inward (depending on the sign of J_r). Since current, if any, has to flow in closed loops, this means there cannot be any radial currents.

Equation 1 now reads $\frac{\partial J_x}{\partial x} = 0$, so J_x must be a function of r only. However, similar to our argument in the previous paragraph, if we have a nonzero J_x at some point (r, θ, x), it will be impossible for said current to flow in a closed loop since all points with the same r will have the same J_x. Hence $J_x = 0$ everywhere inside the sphere.

Finally, we can conclude that $J_\theta = 0$ by realizing that a nonzero J_θ would imply a nonzero tangential component of $\vec{E}$. However, since the charge distribution is symmetric, ρ_{in} cannot depend on θ, and thus the electric potential ϕ inside the sphere also cannot depend on θ. Hence, there cannot be a tangential component of $\vec{E} = -\nabla\phi$, so $J_\theta = 0$. We have hence shown that there is no current density anywhere inside the rotating sphere , and so we finally obtain the electric field inside the sphere:

$$\vec{E}_{in} = -(\vec{v} \times \vec{B}) = -\omega r B \ \hat{r} \tag{2}$$

Hence, the internal charge density is given by

$$\rho_{in} = \epsilon \nabla \cdot \vec{E}_{in} = \epsilon \left(\frac{1}{r}\frac{\partial}{\partial r}(-\omega r^2 B) \right) = -2\omega\epsilon B \tag{3}$$

Interestingly, ρ_{in} does not just have cylindrical symmetry, it is actually uniform everywhere inside the sphere!

4 Outside the Rotating Sphere

Since there is negative volume charge density everywhere inside the sphere, there must be positive surface charge density on the sphere's surface. Of course, this surface charge distribution must have cylindrical symmetry, but it must also be symmetric about the $x = 0$ plane. To see this, imagine cutting the sphere into infinitesimally thin slices parallel to the yz plane. The magnetic field and enclosed internal charge are the same for the slice at $x = A$ and the slice at

$x = -A$, where $|A| \leq R$. Hence the surface charge distribution must be symmetric about the $x = 0$ plane.

Qualitatively, since the sphere's total charge (inside charges + outside charges) must be zero, the sphere must have some kind of internal polarization in a sense. If we cut the sphere into infinitesimal slices parallel to the yz plane, the slices at the two ends of the sphere ($x = \pm R$) should end up with a net charge of some sort, while the slice at the centre of the sphere ($x = 0$) should have a net charge of the opposite sign. This motivates a hypothesis that the electric field outside the sphere is actually a linear quadrupole field! Of course, by symmetry the quadrupole must be located at the origin. We now need to find the electric field everywhere and ensure it satisfies the boundary conditions on the sphere's surface, namely:

1. The component of the electric field tangential to the sphere's surface is continuous. This follows from Maxwell's equation $\nabla \times \vec{E} = -\frac{\partial B}{\partial t} = 0$.
2. The total surface charge at the sphere's boundary (which can be computed via Gauss' Law) plus the negative internal charge must be zero. In other words, the total surface charge must be $2\omega\epsilon B\left(\frac{4}{3}\pi R^3\right) = \frac{8}{3}\pi\omega\epsilon B R^3$.

As long as we can find the electric field everywhere and this field satisfies the boundary conditions, then it is the only valid field. This follows from the uniqueness of solutions to Laplace's equation (See Section 6 for a proof). In our case, outside the sphere the potential V satisfies $\nabla^2 V = 0$ and the potential at the sphere's boundary can be determined from the internal electric field, with an additive constant for $V(0,0,0)$ that can be determined from the 2 aforementioned boundary conditions. Hence the conditions for the uniqueness of the solution are satisfied. If V is unique, then $\vec{E}$ is too.

To begin, let us find the electric field due to a linear quadrupole by treating it as a system of balls of charge Q, $-2Q$ and Q with a small distance d between adjacent charges. The configuration is illustrated in Figure 2.

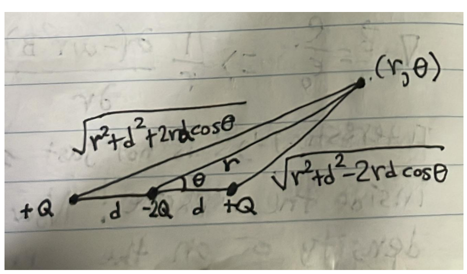

Figure 2: We will evaluate the potential at a point (r, θ).

We will work in polar coordinates due to symmetry about the x-axis. The electric potential at a point (r, θ), where $r >> d$, is given by

$$V = \frac{1}{4\pi\epsilon_0}\left[-\frac{2Q}{r} + \frac{Q}{r}\left(1 + \frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right)^{-\frac{1}{2}} + \frac{Q}{r}\left(1 + \frac{d^2}{r^2} + \frac{2d}{r}\cos\theta\right)^{-\frac{1}{2}}\right]$$

Carrying out a Taylor expansion and taking the leading order terms gives us

$$V \approx \frac{Q}{4\pi\epsilon_0 r}\left[-2 + \left(1 - \frac{1}{2}\left(\frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right)^2\right)\right]$$

$$+\frac{Q}{4\pi\epsilon_0 r}\left[\left(1 - \frac{1}{2}\left(\frac{d^2}{r^2} + \frac{2d}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{d^2}{r^2} + \frac{2d}{r}\cos\theta\right)^2\right)\right]$$

$$\approx \frac{Q}{4\pi\epsilon_0 r}\left[-\frac{d^2}{r^2} + 2 \times \frac{3}{8}\left(\frac{4d^2}{r^2}\cos^2\theta\right)\right] = \frac{Qd^2}{4\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right)$$

Now, we compute the electric field outside the sphere due to the quadrupole. In the polar coordinates that we are using,

$$\vec{E}_{out} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$\vec{E}_{out} = \frac{3Qd^2}{4\pi\epsilon_0 r^4}\left(3\cos^2\theta - 1\right)\hat{r} + \frac{3Qd^2\sin\theta\cos\theta}{2\pi\epsilon_0 r^4}\hat{\theta}$$

Let's define $Qd^2 = p$ as the quadrupole moment. Then we can re-express the above equation as

$$\vec{E}_{out} = \frac{3p}{4\pi\epsilon_0 r^4}\left(3\cos^2\theta - 1\right)\hat{r} + \frac{3p\sin\theta\cos\theta}{2\pi\epsilon_0 r^4}\hat{\theta} \tag{4}$$

Now we can start verifying the boundary conditions. To do so, let us express the electric field inside the sphere in terms of our polar coordinates:

$$\vec{E}_{in} = -\omega r B\sin\theta(\sin\theta\hat{r} + \cos\theta\hat{\theta})$$

Continuity of the tangential component means that, at $r = R$, we have

$$-\omega R B\sin\theta\cos\theta = \frac{3p\sin\theta\cos\theta}{2\pi\epsilon_0 R^4} \implies p = -\frac{2}{3}\pi\epsilon_0\omega B R^5 \tag{5}$$

Notably, our value for p is negative, which means that the ends of the sphere (at $x = \pm R$) actually have a net negative charge and the centre of the sphere (at $x = 0$) actually has a net positive charge. As a sanity check, flipping the direction of either ω or B also flips the charge distribution, changing the polarity of our quadrupole, so the current equation matches physical intuition.

Now let us verify the second boundary condition. Using an infinitely small Gaussian cylinder at the sphere's boundary oriented radially, we obtain the surface charge density σ on the sphere's surface:

$$\epsilon_0 E_{r,out} - \epsilon E_{r,in} = \sigma \implies \sigma = \frac{3p}{4\pi R^4}\left(3\cos^2\theta - 1\right) + \epsilon\omega RB\sin^2\theta$$

Integrating this over the surface of the sphere gives a total surface charge of

$$Q_{surface} = \int_0^\pi \sigma\,(2\pi R\sin\theta)\,(R\ \mathrm{d}\theta)$$

$$= 2\pi R^2 \int_0^\pi \frac{3p}{4\pi R^4}\left(3\cos^2\theta - 1\right)(\sin\theta)\ \mathrm{d}\theta + 2\pi\epsilon\omega R^3 B\int_0^\pi \sin^3\theta\ \mathrm{d}\theta$$

$$= \frac{8}{3}\pi\omega\epsilon BR^3$$

This exactly cancels out the negative charge inside the sphere, and thus the second boundary condition is satisfied, no matter what the value of ϵ is. This is therefore the only valid solution for the electric field.

There is one small problem - the surface charges are moving, which can result in a magnetic field inside the sphere that differs from B. However, we can show that the magnetic field generated by the moving charges is negligible using an order of magnitude estimate. The effective current is $I \sim \sigma R(\omega R) \sim \omega^2 BR^3\epsilon_0$, and hence, the additional field would be $B_{additional} \sim \frac{\mu_0 I}{R} \sim \mu_0\epsilon_0\omega^2 BR^2 = \frac{(\omega R)^2}{c^2}B$, which is negligible. Hence we can still assume the magnetic field in the sphere is B.

5 Interaction Force

Due to the quadrupole field generated by the sphere at $x = 0$, some charge distribution will be induced on the conducting sphere at $x = L$ in order for it to remain an equipotential surface. This allows us to use the method of images to evaluate the interaction force (See Section 7 for a proof).

There are 4 image charges in the sphere (we will ignore second-order image charges since $L >> R$):

1. A charge $-Q\frac{R}{L-d}$ at $x = L - \frac{R^2}{L-d}$.
2. A charge $2Q\frac{R}{L}$ at $x = L - \frac{R^2}{L}$.
3. A charge $-Q\frac{R}{L+d}$ at $x = L - \frac{R^2}{L+d}$.

4. A charge $Q\left(\frac{R}{L-d}+\frac{R}{L+d}-2\frac{R}{L}\right)\approx\frac{2Rd^2}{L^3}Q$ at $x=L$ to ensure that the conductor has zero net charge.

Before we evaluate the interaction force, let us consider the force on the quadrupole due to a nonuniform electric field. Consider our original quadrupole, with charges Q at $x=\pm d$ and charge $-2Q$ at $x=0$. We subject it to a nonuniform electric field $E(x)\hat{x}$. Then the force on the quadrupole is

$$\vec{F}=Q\left[E(-d)-2E(0)+E(d)\right]\hat{x}=Qd\left[\frac{E(d)-E(0)}{d}-\frac{E(0)-E(-d)}{d}\right]\hat{x}$$

As we take the limit $d\to 0$ we obtain

$$\vec{F}=Qd\left[E'(0)-E'(-d)\right]\hat{x}=Qd^2E''(-d)\ \hat{x}=pE''(0)\ \hat{x}$$

The field due to the charges can be found by using Coulomb's Law and expanding to the leading order. We know that there is no net charge on the sphere, so there should not be an inverse square field. On the other hand, we have some positive charge at $x=L$ and a net negative charge centered around $x=L-\frac{R^2}{L}$ - a dipole moment. Hence the field should follow an inverse cube law (to leading order). At $x=L-r$, $r>0$, the field due to the image charges is

$$\vec{E}_{image}=\frac{1}{4\pi\epsilon_0}\left[\frac{Q\frac{R}{L-d}}{\left(r-\frac{R^2}{L-d}\right)^2}+\frac{Q\frac{R}{L+d}}{\left(r-\frac{R^2}{L+d}\right)^2}-\frac{2Q\frac{R}{L}}{\left(r-\frac{R^2}{L}\right)^2}-\frac{Q\frac{2Rd^2}{L^3}}{r^2}\right]\hat{x}$$

$$=\frac{QR}{4\pi\epsilon_0Lr^2}\left[\left(1-\frac{d}{L}\right)^{-1}\left(1-\frac{R^2}{(L-d)r}\right)^{-2}+\left(1+\frac{d}{L}\right)^{-1}\left(1-\frac{R^2}{(L+d)r}\right)^{-2}-2\left(1-\frac{R^2}{Lr}\right)^{-2}-2\frac{d^2}{L^2}\right]\hat{x}$$

$$\approx\frac{QR}{4\pi\epsilon_0Lr^2}\left[\left(1+\frac{d}{L}+\frac{d^2}{L^2}\right)\left(1+\frac{2R^2}{(L-d)r}\right)+\left(1-\frac{d}{L}+\frac{d^2}{L^2}\right)\left(1+\frac{2R^2}{(L+d)r}\right)-2\left(1+\frac{2R^2}{Lr}\right)-2\frac{d^2}{L^2}\right]\hat{x}$$

$$=\frac{QR}{4\pi\epsilon_0Lr^2}\left[\frac{2R^2}{r}\left(\frac{1}{L-d}+\frac{1}{L+d}-\frac{2}{L}\right)+\frac{2R^2d}{Lr}\left(\frac{1}{L-d}-\frac{1}{L+d}\right)+\frac{2R^2d^2}{L^2r}\left(\frac{1}{L-d}+\frac{1}{L+d}\right)\right]\hat{x}$$

$$\approx\frac{QR}{4\pi\epsilon_0Lr^2}\left[\frac{2R^2}{r}\left(\frac{2d^2}{L^3}\right)+\frac{2R^2d}{Lr}\left(\frac{2d}{L^2}\right)+\frac{2R^2d^2}{L^2r}\left(\frac{2}{L}\right)\right]\hat{x}$$

$$=\frac{QR}{4\pi\epsilon_0Lr^2}\left[\frac{12R^2d^2}{L^3r}\right]\hat{x}=\frac{3Qd^2R^3}{\pi\epsilon_0L^4r^3}\hat{x}=\frac{3pR^3}{\pi\epsilon_0L^4\left(L-x\right)^3}\hat{x}$$

This is indeed a dipole field to leading order. taking the second derivative gives

$$E''_{image}(x) = \frac{36pR^3}{\pi\epsilon_0 L^4 (L-x)^5} \implies E''_{image}(0) = \frac{36pR^3}{\pi\epsilon_0 L^9}$$

From this, we obtain at last

$$\vec{F}_{rotatingsphere} = pE''_{image}(0)\ \hat{x} = \frac{36p^2R^3}{\pi\epsilon_0 L^9}\hat{x} = \frac{36R^3\left(-\frac{2}{3}\pi\epsilon_0\omega BR^5\right)^2}{\pi\epsilon_0 L^9}\hat{x}$$

Simplifying gives us the final answer

$$\boxed{\vec{F}_{rotating\ sphere} = \frac{16\pi\epsilon_0\omega^2B^2R^{13}}{L^9}\hat{x}} \tag{6}$$

The interaction force between the two spheres is attractive and has magnitude $\frac{16\pi\epsilon_0\omega^2B^2R^{13}}{L^9}$.

6 Appendix - The Uniqueness Theorem

Suppose that a given potential function ϕ satisfies the Laplace equation $\left(\nabla^2\phi = 0\right)$ inside a finite, bounded volume V, and suppose that on the surface S of V we know the value of ϕ. We will prove by contradiction that only one solution exists for ϕ. Suppose we have two solutions ϕ_1 and ϕ_2 that satisfy both conditions. Define $\Phi = \phi_1 - \phi_2$. Then Φ is 0 on S and $\nabla^2\Phi = 0$ inside V. Since $|\nabla\Phi| \neq 0$ (for otherwise $\Phi = 0$ everywhere in V, which would be a contradiction), let us evaluate the following integral:

$$\int_V |\nabla\Phi|^2 d^3r = \int_V \nabla\cdot(\Phi\nabla\Phi)\ d^3r - \int_V \nabla^2\Phi\ d^3r$$

The former integral can be expressed as $\oiint_S (\Phi\nabla\Phi)\cdot d\vec{A}$ via the Divergence Theorem, and hence it vanishes on S. The second integral also vanishes since Φ satisfies Laplace's equation in V. Hence, the LHS is nonnegative while the RHS is zero, which means $|\nabla\Phi| = 0$ everywhere in V, a contradiction. Hence there can only be one solution to the potential function ϕ.

7 Appendix - The Method of Images for Spherical Conductors

Consider a grounded spherical conductor centred at $x = 0$ with radius R. Now suppose we have a positive charge Q at $x = L$. We want to find the charge distribution on the sphere such that all points on its surface are at a potential of 0V. Just outside the sphere, Laplace's equation is satisfied as well as there are no charges. Hence if we can find a valid potential function, it must be the correct potential function.

By symmetry, the image charge must be on the x-axis. Suppose it has charge q and is located at $x = d$. The conductor does not matter since it is grounded, so we can just treat it as an arbitrary equipotential surface. Due to the symmetry we need only consider the two-dimensional case. The potential V at a point (x, y) is given by

$$4\pi\epsilon_0 V = \frac{q}{\sqrt{(x-d)^2+y^2}} + \frac{Q}{\sqrt{(x-L)^2+y^2}}$$

Equating this to zero and rearranging gives

$$\frac{q}{\sqrt{(x-d)^2+y^2}} = -\frac{Q}{\sqrt{(x-L)^2+y^2}}$$

$$\implies Q^2\left((x-d)^2+y^2\right) = q^2\left((x-L)^2+y^2\right)$$

Now we expand:

$$x^2\left(Q^2-q^2\right) + x\left(2q^2L - 2Q^2d\right) + y^2\left(Q^2-q^2\right) + Q^2d^2 - q^2L^2 = 0$$

We want to match this with the express for a circle $x^2 + y^2 = R^2$ since the surface with 0 potential needs to be a circle. Comparing coefficients tells us that

$$q = -Q\sqrt{\frac{d}{L}}, \quad \frac{q^2L^2 - Q^2d^2}{Q^2 - q^2} = R^2$$

Of course, q must be opposite in sign to Q for there to be points with zero potential that aren't infinity. Substituting the first equation into the second equation gives

$$\frac{Q^2Ld - Q^2d^2}{Q^2\left(1-\frac{d}{L}\right)} = R^2 \implies L^2d - Ld^2 = R^2(L-d)$$

$$\implies Ld^2 - (L^2+R^2)d + R^2L = 0$$

We solve this to get

$$d = \frac{(L^2+R^2) \pm \sqrt{(L^4+R^4+2R^2L^2-4R^2L^2)}}{2L} = \frac{L^2+R^2 \pm (L^2-R^2)}{2L}$$

The larger root is L, but that corresponds to the charge Q. Hence the x-position of charge q is given by the smaller root $\frac{R^2}{L}$. Thus, we also have $q = -Q\frac{R}{L}$. If the conductor is neutral but not grounded, adding an image charge at the centre of the sphere to make the net charge zero is a valid configuration for the equipotential surface; by uniqueness, it is the only valid configuration.