Physics Cup 2025 - Problem 1

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1 Electromagnetic Field inside Ball A

Let ball A be the rotating metal ball placed at the origin. It rotates with angular velocity $\boldsymbol{\omega} = \omega\hat{\boldsymbol{x}}$ in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{x}}$. The fields, the potentials and the current and charge densities can be calculated as follows[1]:

Let $\sigma_c = \frac{1}{\rho_c}$ be the conductivity of the ball, ρ its charge density and $\boldsymbol{J}$ the current density inside. Now, $\boldsymbol{J} = \sigma_c[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}]$ and the continuity equation says that $\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$. Together with Gauss's law, then:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = \sigma_c[\boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{B})] = \sigma_c \left[\frac{\rho}{\epsilon_0} + \boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{B})\right] = -\frac{\partial \rho}{\partial t}$$

Since the system is in a steady state (instantaneously at least), ρ is constant and thus,

$$\rho = -\epsilon_0 \boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{B})$$

With $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, $\boldsymbol{v} \times \boldsymbol{B} = -\boldsymbol{B} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = -\boldsymbol{\omega}(\boldsymbol{B} \cdot \boldsymbol{r}) + \boldsymbol{r}(\boldsymbol{B} \cdot \boldsymbol{\omega}) = \omega B(\boldsymbol{r} - x\hat{\boldsymbol{x}})$. Then,

$$\rho = -2\epsilon_0 \omega B \tag{1}$$

So, there appears a uniform volume charge density together with a compensating surface charge density σ, which will be calculated later. Poisson's equation gives together with ρ,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} = 2\omega B$$

The radial current has to vanish so that no charge leaks off the ball, so $J_r = \sigma_c[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}] \cdot \hat{\boldsymbol{r}} = 0$, from which it follows that $E_r = -\omega B(\boldsymbol{r} - x\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{r}} = -\frac{\omega B}{r}(y^2 + z^2) = -\omega B r \sin^2\theta$, where θ denotes the angle between $\boldsymbol{r}$ and the x-axis. This provides a boundary condition on V, namely, $\frac{\partial V}{\partial r}|_{r=R} = \omega B R \sin^2\theta$. By the uniqueness theorem, this boundary condition and ρ uniquely determine the electric field. In fact, the following V fits the required conditions as is easily seen by taking the normal derivative and the Laplacian,

$$V_{r<R}(r, \theta) = \frac{\omega B}{2} r^2 \sin^2\theta + V_0 = \frac{\omega B}{2}(y^2 + z^2) + V_0 \tag{2}$$

The electric field can be found from the negative gradient,

$$\boldsymbol{E} = -\boldsymbol{\nabla} V = -\omega B(y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) = -\boldsymbol{v} \times \boldsymbol{B} \tag{3}$$

There are no eddy currents since the contributions from the electric and magnetic fields exactly cancel, so $\boldsymbol{J} = \boldsymbol{0}$. The currents due to the rotational motion of the volume and surface charges are negligible[2].

[1] David J. Griffiths, *Introduction to Electrodynamics*, Fifth Edition (2023), Example 7.5 p.312

[2] See appendix: Motivation for Approximations

2 Electromagnetic Field produced by Ball A outside its boundaries

Ignoring for a moment ball B at $x = L$, the space outside ball A is empty and thus, Laplace's equation holds there. Let us first show that the potential in this region can be expanded in terms of Legendre polynomials as follows[3]:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{4}$$

Since the problem has azimuthal symmetry, V only depends on r and θ, so Laplace's equation reduces to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta} \right) = 0 \tag{5}$$

One can try looking for separable solutions of the form $V(r,\theta) = R(r)\Theta(\theta)$. Putting this V into Eq.(5) and dividing by V yields

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r} \right) + \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta} \right) = 0$$

Since each one of the two terms only depends on one of both variables r and θ, both must be a constant and its opposite respectively. The constant is most conveniently written as $l(l+1)$. The radial equation $\frac{\mathrm{d}}{\mathrm{d}r}(r^2\frac{\mathrm{d}R}{\mathrm{d}r}) = l(l+1)R$ has as solution $R(r) = Ar^l + \frac{B}{r^{l+1}}$ with two constants A and B, as can be checked by plugging $R(r)$ back into the equation. The solutions to the angular equation $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}) = -l(l+1)\sin\theta\Theta$ are Legendre polynomials in the variable $\cos\theta$, the three first of whom are listed below:

$$\begin{aligned} P_0(\cos\theta) &= 1 \\ P_1(\cos\theta) &= \cos\theta \\ P_2(\cos\theta) &= \frac{3\cos^2\theta - 1}{2} = 1 - \frac{3}{2}\sin^2\theta \end{aligned}$$

The second solution to the second order differential angular equation blows up at $\theta = 0$ and/or at $\theta = \pi$ and can therefore be rejected. A linear combination of $R(r)\Theta(\theta)$ over all values of l gives Eq.(4).

Now, A_l and B_l have to be determined by matching Eq.(4) to Eq.(2) at the surface of ball A[4]. First of all, $V \to 0$ as $r \to \infty$, so $A_l = 0$. It works out with $B_l = 0$ except for $l = 0$ and $l = 2$ and, by the uniqueness theorem, this gives the only right potential. So,

$$V_{r>R}(r,\theta) = \frac{B_0}{r} + \frac{B_2}{r^3}\left(1 - \frac{3}{2}\sin^2\theta \right)$$

Comparing with Eq.(2) at $r = R$, $B_2 = -\frac{\omega BR^5}{3}$ and $\frac{B_0}{R} - \frac{\omega BR^2}{3} = V_0$ and thus $B_0 = V_0 R + \frac{\omega BR^3}{3}$.

$$V_{r>R}(r,\theta) = V_0\frac{R}{r} + \frac{\omega B}{3}\left(\frac{R^3}{r} - \frac{R^5}{r^3} + \frac{3R^5}{2r^3}\sin^2\theta \right) \tag{6}$$

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[3] David J. Griffiths, *Introduction to Electrodynamics*, Fifth Edition (2023), Section 3.3.2 p.139

[4] Kirk T. McDonald, *Conducting Sphere That Rotates in a Uniform Magnetic Field*, Princeton University (2002)

By determining the total surface charge from Eq.(6) and Eq.(2) and by equating it to the opposite of the total volume charge (the ball is neutral as a whole), V_0 can be determined. Using the boundary condition of an electric field at a surface charge (Gaussian pillbox),

$$\frac{\partial V_{r>R}}{\partial r}\Big|_{r=R} - \frac{\partial V_{r<R}}{\partial r}\Big|_{r=R} = -\frac{\sigma}{\epsilon_0}$$

The derivatives are:

$$\frac{\partial V_{r>R}}{\partial r}\Big|_{r=R} = -\frac{V_0}{R} + \frac{\omega B}{3}\left(-R + 3R - 3\frac{3R}{2}\sin^2\theta\right) = -\frac{V_0}{R} + \frac{\omega BR}{3}\left(2 - \frac{9}{2}\sin^2\theta\right)$$
$$\frac{\partial V_{r<R}}{\partial r}\Big|_{r=R} = \omega BR\sin^2\theta$$

Thus,

$$-\frac{V_0}{R} + \frac{\omega BR}{3}\left(2 - \frac{9}{2}\sin^2\theta\right) - \omega BR\sin^2\theta = -\frac{\sigma}{\epsilon_0}$$
$$\sigma = \frac{\epsilon_0 V_0}{R} + \epsilon_0\omega BR\left(\frac{5}{2}\sin^2\theta - \frac{2}{3}\right) \tag{7}$$

Integrating over the surface of the ball gives the total surface charge:

$$Q_S = \frac{\epsilon_0 V_0}{R}\cdot 4\pi R^2 + \epsilon_0\omega BR\cdot 2\pi R^2\left(\frac{5}{2}\cdot\frac{4}{3} - \frac{2}{3}\cdot 2\right)$$
$$= 4\pi\epsilon_0 V_0 R + 4\pi\epsilon_0\omega BR^3$$

The total volume charge is $Q_V = -2\epsilon_0\omega B\cdot\frac{4}{3}\pi R^3 = -\frac{8}{3}\pi\epsilon_0\omega BR^3$. The ball is neutral, so $Q_S = -Q_V$ which gives $V_0 = -\frac{\omega BR^2}{3}$. This allows to find the potential outside the ball as well as its surface charge density,

$$V_{r>R}(r,\theta) = -\frac{\omega BR^3}{3r} + \frac{\omega B}{3}\left(\frac{R^3}{r} - \frac{R^5}{r^3} + \frac{3R^5}{2r^3}\sin^2\theta\right) = -\frac{\omega BR^5}{3r^3}\left(1 - \frac{3}{2}\sin^2\theta\right) \tag{8}$$

$$\sigma = \epsilon_0\omega BR\left(\frac{5}{2}\sin^2\theta - 1\right) \tag{9}$$

Finally, taking the opposite of the gradient of the potential yields the electric field outside ball A:

$$\begin{aligned} E_{r,r>R} &= -\frac{\omega BR^5}{r^4}\left(1 - \frac{3}{2}\sin^2\theta\right) \\ E_{\theta,r>R} &= -\frac{\omega BR^5}{2r^4}\sin(2\theta) \end{aligned} \tag{10}$$

3 Equivalent Charge Configuration in Ball A

The electric field created by ρ and σ induces an image charge configuration in ball B which is placed at $x = L \gg R$. To simplify calculations, it is worth noting that ρ and the θ-independent part of σ are equivalent to an effective point charge q_V placed at the centre of ball A, by their angular symmetry and by Gauss's law. Furthermore, the $\sin^2\theta$ term in σ describes a charge accumulation around the

y-z-plane passing through the centre of ball A. Therefore, for points far from ball A, this term can be approximated by a charged ring of radius $\gamma R < R$ in the same plane, its centre coinciding with the centre of ball A.

Integrating the θ-independent term of σ over the surface of ball A and adding Q_V gives q_V,

$$q_V = -\frac{8}{3}\pi\epsilon_0\omega BR^3 - \epsilon_0\omega BR \cdot 4\pi R^2 = -\frac{20}{3}\pi\epsilon_0\omega BR^3 \tag{11}$$

The total charge of the ring turns of course out to be $-q_V$ by the same integration over the surface,

$$q_S = \frac{5}{2}\epsilon_0\omega BR \cdot 2\pi R^2 \cdot \frac{4}{3} = \frac{20}{3}\pi\epsilon_0\omega BR^3 \tag{12}$$

The electric field of the charged ring on the positive x-axis is easily determined since it has only an x-component because of symmetry:

$$\begin{aligned}\boldsymbol{E}_{q_S} &= \frac{q_S}{4\pi\epsilon_0} \cdot \frac{1}{x^2 + (\gamma R)^2} \cdot \cos\theta' \hat{\boldsymbol{x}} \\ &= \frac{q_S}{4\pi\epsilon_0} \cdot \frac{x}{[x^2 + (\gamma R)^2]^{\frac{3}{2}}} \hat{\boldsymbol{x}}\end{aligned}$$

The combined field of q_V and q_S on the positive x-axis is,

$$\begin{aligned}\boldsymbol{E}_{q_V,q_S} &= \frac{q_V}{4\pi\epsilon_0} \cdot \frac{1}{x^2}\hat{\boldsymbol{x}} + \frac{q_S}{4\pi\epsilon_0} \cdot \frac{x}{[x^2 + (\gamma R)^2]^{\frac{3}{2}}} \hat{\boldsymbol{x}} \\ &= \frac{5\omega BR^3}{3}\left[-\frac{1}{x^2} + \frac{x}{[x^2 + (\gamma R)^2]^{\frac{3}{2}}}\right]\hat{\boldsymbol{x}}\end{aligned} \tag{13}$$

Expanding this result to first order in powers of $\left(\frac{\gamma R}{x}\right)^2$ reveals that for large x,

$$\begin{aligned}\boldsymbol{E}_{q_V,q_S} &\approx \frac{5\omega BR^3}{3}\left[-\frac{1}{x^2} + \frac{1}{x^2} - \frac{1}{x^2} \cdot \frac{3}{2}\left(\frac{\gamma R}{x}\right)^2\right]\hat{\boldsymbol{x}} \\ &= -\frac{5\omega BR^5}{2x^4}\gamma^2 \hat{\boldsymbol{x}}\end{aligned}$$

Comparing this result to $E_{r,r>R}$ in Eq.(10) with $\theta = 0$ gives $\gamma = \sqrt{\frac{2}{5}}$.

4 Image of a Charge outside a Conducting Ball

First consider the following general calculation of the image charge q' appearing due to a charge q placed a distance a from the centre of a grounded conducting ball of radius R at the origin. (Figure 1) The potential created by this configuration is $V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{d} + \frac{q'}{d'}\right)$. For points on the surface, $V = 0$. In particular, at $\boldsymbol{r} = -R\hat{\boldsymbol{x}}$ and at $\boldsymbol{r} = R\hat{\boldsymbol{x}}$,

$$\begin{aligned}V(-R\hat{\boldsymbol{x}}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a+R} + \frac{q'}{b+R}\right) = 0 \\ V(R\hat{\boldsymbol{x}}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a-R} + \frac{q'}{R-b}\right) = 0\end{aligned}$$

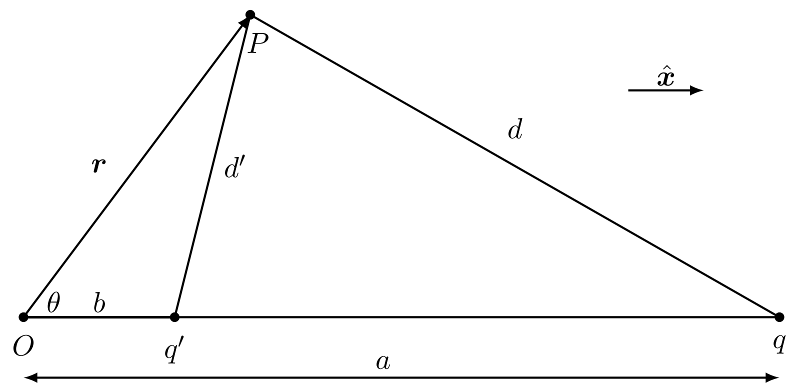

Figure 1: *Image Charge q' in Conducting Grounded Ball due to Charge q outside*

This system of equations can easily be solved to get b and q':

$$\begin{aligned} b &= \frac{R^2}{a} \\ q' &= -\frac{R}{a}q \end{aligned} \tag{14}$$

With

$$\begin{aligned} d &= \sqrt{R^2 + a^2 - 2Ra\cos\theta} \\ d' &= \sqrt{R^2 + b^2 - 2Rb\cos\theta} = \sqrt{R^2 + \frac{R^4}{a^2} - 2\frac{R^3}{a}\cos\theta} = \frac{R}{a}\sqrt{a^2 + R^2 - 2Ra\cos\theta} \end{aligned}$$

the potential at the surface of the ball is $V(R) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{R^2+a^2-2Ra\cos\theta}} + \frac{-\frac{R}{a}q}{\frac{R}{a}\sqrt{a^2+R^2-2Ra\cos\theta}}\right) \overset{!}{=} 0$ which confirms by the uniqueness theorem that Eq.(14) gives the right parameters.

If the ball is not grounded, charge has to be conserved. Therefore, an additional point charge has to be placed at the centre of the ball to make it neutral again. Since the field of the charge is radial, it doesn't destroy the boundary condition at the surface of the ball.

5 Induced Image Charge Configuration in Ball B

From Eq.(14), it follows that the image charge $q_{V_{\text{Im}}}$ of q_V is placed at $x_{V_{\text{Im}}} = L - \Delta x_{V_{\text{Im}}}$ where,

$$\begin{aligned} q_{V_{\text{Im}}} &= -\frac{R}{L}q_V = \frac{20}{3}\pi\epsilon_0\omega B\frac{R^4}{L} \\ \Delta x_{V_{\text{Im}}} &= \frac{R^2}{L} \end{aligned} \tag{15}$$

Since, by symmetry, each piece of charge of q_S is subject to the same transformation to get to its image, the total transforms in the same way. The image of the charged ring is again a charged ring of radius $\gamma' R$ which lies in the y-z-plane and whose centre is placed at $x_{S_{\rm Im}} = L - \Delta x_{S_{\rm Im}}$.

$$\begin{aligned} q_{S_{\rm Im}} &= -\frac{R}{\sqrt{L^2+(\gamma R)^2}} q_S = -\frac{20}{3}\pi\epsilon_0 \omega B \frac{R^4}{\sqrt{L^2+(\gamma R)^2}} \\ \Delta x_{S_{\rm Im}} &= \frac{R^2}{\sqrt{L^2+(\gamma R)^2}} \cdot \cos\theta' = \frac{LR^2}{L^2+(\gamma R)^2} \\ \gamma' &= \frac{R}{\sqrt{L^2+(\gamma R)^2}} \cdot \sin\theta' = \frac{R^2}{L^2+(\gamma R)^2} \ll 1 \end{aligned} \tag{16}$$

Since the initially neutral ball B is not grounded, it has to stay neutral by conservation of charge. Therefore, an additional image charge $q_{C_{\rm Im}}$ has to be placed at its centre at $x_{C_{\rm Im}} = L$ with,

$$\begin{aligned} q_{C_{\rm Im}} &= -(q_{V_{\rm Im}} + q_{S_{\rm Im}}) \\ &= \frac{20}{3}\pi\epsilon_0\omega B R^4 \left[\frac{1}{\sqrt{L^2+(\gamma R)^2}} - \frac{1}{L}\right] \end{aligned} \tag{17}$$

As is shown in the appendix, the effect of these image charges on the mechanism creating ρ and σ is negligible.

6 Interaction Force between both Balls

The approach is to expand the force on $q_{V_{\rm Im}}$, $q_{S_{\rm Im}}$ and $q_{C_{\rm Im}}$ in powers of $\frac{R}{L}$ and to keep only the lowest order term since $R \ll L$. The combined field of q_V and q_S on the x-axis is given by Eq.(13). This expression can also be used for $q_{S_{\rm Im}}$ since $\gamma' \ll 1$, which is equal to treating it as a point charge.
The force on $q_{C_{\rm Im}}$ has the following magnitude,

$$\begin{aligned} F_{C_{\rm Im}} &= \frac{20}{3}\pi\epsilon_0\omega B R^4 \left[\frac{1}{\sqrt{L^2+(\gamma R)^2}} - \frac{1}{L}\right] \cdot \frac{5\omega B R^3}{3} \left[-\frac{1}{L^2} + \frac{L}{[L^2+(\gamma R)^2]^{\frac{3}{2}}}\right] \\ &= \frac{100}{9}\pi\epsilon_0\omega^2 B^2 \frac{R^7}{L^3} \left[-\frac{1}{2}\left(\frac{\gamma R}{L}\right)^2 + \frac{3}{8}\left(\frac{\gamma R}{L}\right)^4 - \ldots\right] \\ &\quad \left[-\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2 + \frac{15}{8}\left(\frac{\gamma R}{L}\right)^4 - \ldots\right] \\ &= \frac{100}{9}\pi\epsilon_0\omega^2 B^2 \frac{R^7}{L^3} \left[\frac{3}{4}\left(\frac{\gamma R}{L}\right)^4 - \frac{15}{16}\left(\frac{\gamma R}{L}\right)^6 - \frac{9}{16}\left(\frac{\gamma R}{L}\right)^6 + \ldots\right] \\ &= \frac{4\pi\epsilon_0}{3} \cdot \frac{\omega^2 B^2 R^{11}}{L^7} - \frac{16\pi\epsilon_0}{15} \cdot \frac{\omega^2 B^2 R^{13}}{L^9} + \ldots \end{aligned}$$

The force on $q_{V_{\rm Im}}$ has the magnitude,

$$F_{V_{\rm Im}} = \frac{20}{3}\pi\epsilon_0\omega B \frac{R^4}{L} \cdot \frac{5\omega B R^3}{3} \left[-\frac{1}{x_{V_{\rm Im}}^2} + \frac{x_{V_{\rm Im}}}{\left[x_{V_{\rm Im}}^2 + (\gamma R)^2\right]^{\frac{3}{2}}}\right]$$

$$= \frac{100}{9}\pi\epsilon_0\omega^2 B^2 \frac{R^7}{L}\left[-\frac{1}{x_{V_{\rm Im}}^2}+\frac{1}{x_{V_{\rm Im}}^2}-\frac{1}{x_{V_{\rm Im}}^2}\cdot\frac{3}{2}\left(\frac{\gamma R}{x_{V_{\rm Im}}}\right)^2+\frac{1}{x_{V_{\rm Im}}^2}\cdot\frac{15}{8}\left(\frac{\gamma R}{x_{V_{\rm Im}}}\right)^4-\frac{1}{x_{V_{\rm Im}}^2}\cdot\frac{35}{16}\left(\frac{\gamma R}{x_{V_{\rm Im}}}\right)^6+\ ...\right]$$
$$= \frac{100}{9}\pi\epsilon_0\omega^2 B^2 \frac{R^7}{L^3}\left[-\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\left(1+4\left(\frac{R}{L}\right)^2+10\left(\frac{R}{L}\right)^4+\ ...\right)+\frac{15}{8}\left(\frac{\gamma R}{L}\right)^4\left(1+6\left(\frac{R}{L}\right)^2+\ ...\right)\right.$$
$$\left.-\frac{35}{16}\left(\frac{\gamma R}{L}\right)^6(1+\ ...\ )+\ ...\right]$$
$$= \frac{100}{9}\pi\epsilon_0\omega^2 B^2 \frac{R^7}{L^3}\left[-\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2-6\gamma^2\left(\frac{R}{L}\right)^4+\frac{15}{8}\left(\frac{\gamma R}{L}\right)^4-15\gamma^2\left(\frac{R}{L}\right)^6+\frac{45}{4}\gamma^4\left(\frac{R}{L}\right)^6-\frac{35}{16}\left(\frac{\gamma R}{L}\right)^6-\ ...\right]$$
$$= -\frac{20\pi\epsilon_0}{3}\cdot\frac{\omega^2 B^2 R^9}{L^5}-\frac{70\pi\epsilon_0}{3}\cdot\frac{\omega^2 B^2 R^{11}}{L^7}-\frac{434\pi\epsilon_0}{9}\cdot\frac{\omega^2 B^2 R^{13}}{L^9}-\ ...$$

Finally, the force on $q_{S_{\rm Im}}$ is of magnitude,

$$F_{S_{\rm Im}} = -\frac{20}{3}\pi\epsilon_0\omega B\frac{R^4}{\sqrt{L^2+(\gamma R)^2}}\cdot\frac{5\omega B R^3}{3}\left[-\frac{1}{x_{S_{\rm Im}}^2}+\frac{x_{S_{\rm Im}}}{\left[x_{S_{\rm Im}}^2+(\gamma R)^2\right]^{\frac{3}{2}}}\right]$$
$$= -\frac{100}{9}\pi\epsilon_0\omega^2 B^2\frac{R^7}{L}\left[1-\frac{1}{2}\left(\frac{\gamma R}{L}\right)^2+\frac{3}{8}\left(\frac{\gamma R}{L}\right)^4-\ ...\right]$$
$$\left[-\frac{1}{x_{S_{\rm Im}}^2}+\frac{1}{x_{S_{\rm Im}}^2}-\frac{1}{x_{S_{\rm Im}}^2}\cdot\frac{3}{2}\left(\frac{\gamma R}{x_{S_{\rm Im}}}\right)^2+\frac{1}{x_{S_{\rm Im}}^2}\cdot\frac{15}{8}\left(\frac{\gamma R}{x_{S_{\rm Im}}}\right)^4-\frac{1}{x_{S_{\rm Im}}^2}\cdot\frac{35}{16}\left(\frac{\gamma R}{x_{S_{\rm Im}}}\right)^6+\ ...\right]$$
$$= -\frac{100}{9}\pi\epsilon_0\omega^2 B^2\frac{R^7}{L^3}\left[1-\frac{1}{2}\left(\frac{\gamma R}{L}\right)^2+\frac{3}{8}\left(\frac{\gamma R}{L}\right)^4-\ ...\right]$$
$$\left[-\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\left(1+4\frac{\Delta x_{S_{\rm Im}}}{L}+10\left(\frac{\Delta x_{S_{\rm Im}}}{L}\right)^2+\ ...\right)+\frac{15}{8}\left(\frac{\gamma R}{L}\right)^4\left(1+6\frac{\Delta x_{S_{\rm Im}}}{L}+\ ...\right)\right.$$
$$\left.-\frac{35}{16}\left(\frac{\gamma R}{L}\right)^6(1+\ ...\ )+\ ...\right]$$
$$= -\frac{100}{9}\pi\epsilon_0\omega^2 B^2\frac{R^7}{L^3}\left[1-\frac{1}{2}\left(\frac{\gamma R}{L}\right)^2+\frac{3}{8}\left(\frac{\gamma R}{L}\right)^4-\ ...\right]$$
$$\left[-\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\left(1+4\left(\frac{R}{L}\right)^2\left(1-\left(\frac{\gamma R}{L}\right)^2+\ ...\right)+10\left(\frac{R}{L}\right)^4(1-\ ...\ )+\ ...\right)\right.$$
$$+\frac{15}{8}\left(\frac{\gamma R}{L}\right)^4\left(1+6\left(\frac{R}{L}\right)^2(1-\ ...\ )+\ ...\right)$$
$$\left.-\frac{35}{16}\left(\frac{\gamma R}{L}\right)^6(1+\ ...\ )+\ ...\right]$$

$$= -\frac{100}{9}\pi\epsilon_0\omega^2B^2\frac{R^7}{L^3}\left[-\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2 - \frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\cdot 4\left(\frac{R}{L}\right)^2 + \frac{15}{8}\left(\frac{\gamma R}{L}\right)^4 + \frac{1}{2}\left(\frac{\gamma R}{L}\right)^2\cdot\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\right.$$
$$+\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\cdot 4\left(\frac{R}{L}\right)^2\cdot\left(\frac{\gamma R}{L}\right)^2 - \frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\cdot 10\left(\frac{R}{L}\right)^4 + \frac{15}{8}\left(\frac{\gamma R}{L}\right)^4\cdot 6\left(\frac{R}{L}\right)^2 - \frac{35}{16}\left(\frac{\gamma R}{L}\right)^6$$
$$\left.+\frac{1}{2}\left(\frac{\gamma R}{L}\right)^2\cdot\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2\cdot 4\left(\frac{R}{L}\right)^2 - \frac{1}{2}\left(\frac{\gamma R}{L}\right)^2\cdot\frac{15}{8}\left(\frac{\gamma R}{L}\right)^4 - \frac{3}{8}\left(\frac{\gamma R}{L}\right)^4\cdot\frac{3}{2}\left(\frac{\gamma R}{L}\right)^2 - \ ...\right]$$
$$= \frac{20\pi\epsilon_0}{3}\cdot\frac{\omega^2B^2R^9}{L^5} + 22\pi\epsilon_0\cdot\frac{\omega^2B^2R^{11}}{L^7} + \frac{1498\pi\epsilon_0}{45}\cdot\frac{\omega^2B^2R^{13}}{L^9} + \ ...$$

In the appendix, it is shown that the force due to image charges induced in ball A by $q_{V_{\mathrm{Im}}}$, $q_{S_{\mathrm{Im}}}$ and $q_{C_{\mathrm{Im}}}$ is negligible since $R \ll L$. Therefore, the force on ball B is,

$$\begin{aligned}
\boldsymbol{F}_B &= \boldsymbol{F}_{C_{\mathrm{Im}}} + \boldsymbol{F}_{V_{\mathrm{Im}}} + \boldsymbol{F}_{S_{\mathrm{Im}}} \\
&= \left(\frac{4\pi\epsilon_0}{3}\cdot\frac{\omega^2B^2R^{11}}{L^7} - \frac{16\pi\epsilon_0}{15}\cdot\frac{\omega^2B^2R^{13}}{L^9} + \ ...\right. \\
&\quad - \frac{20\pi\epsilon_0}{3}\cdot\frac{\omega^2B^2R^9}{L^5} - \frac{70\pi\epsilon_0}{3}\cdot\frac{\omega^2B^2R^{11}}{L^7} - \frac{434\pi\epsilon_0}{9}\cdot\frac{\omega^2B^2R^{13}}{L^9} - \ ... \\
&\quad \left.+\frac{20\pi\epsilon_0}{3}\cdot\frac{\omega^2B^2R^9}{L^5} + 22\pi\epsilon_0\cdot\frac{\omega^2B^2R^{11}}{L^7} + \frac{1498\pi\epsilon_0}{45}\cdot\frac{\omega^2B^2R^{13}}{L^9} + \ ...\right)\hat{\boldsymbol{x}} \\
&= -16\pi\epsilon_0\frac{\omega^2B^2R^{13}}{L^9}\hat{\boldsymbol{x}}
\end{aligned}$$

So, the interaction force between both balls acts along the x-axis, is attractive and has the magnitude,

$$\boxed{F = 16\pi\epsilon_0\frac{\omega^2B^2R^{13}}{L^9}} \qquad \text{(attractive)} \tag{18}$$

7 Appendix: Calculation of the Interaction Force based on Dipole Induction

Ball B is placed at $x = L \gg R$, so for points close to or inside Ball B, $\sin\theta \approx 0$ and variations in r of the order of R can be neglected. Therefore, the electric field of ball A can be assumed to be uniform at ball B. Calling this uniform field $\boldsymbol{E}_{A,0}$, from Eq.(10),

$$\boldsymbol{E}_{A,0} = -E_{A,0}\hat{\boldsymbol{x}} = -\frac{\omega B R^5}{L^4}\hat{\boldsymbol{x}} \tag{19}$$

Ball B is a conductor at rest, its surface is an equipotential and thus, charges will be induced on it in order to keep the potential constant over the surface. The influence of the surface charges on the outer field can be modelled by placing a point dipole $\boldsymbol{p}_B = -p_B\hat{\boldsymbol{x}}$ in its centre. Let r' denote the distance from the centre of ball B and let $V' = 0$ in the y-z-plane passing through the centre of ball B for points close to it. Then, the potential is

$$V'(r', \theta') = x'E_{A,0} - \frac{1}{4\pi\epsilon_0}\frac{p_B\cos\theta'}{r'^2} = \cos\theta'\left(r'E_{A,0} - \frac{1}{4\pi\epsilon_0}\frac{p_B}{r'^2}\right)$$

Since the surface of the ball is an equipotential, $V'(r' = R, \theta') = 0$, so

$$p_B = 4\pi\epsilon_0 R^3 E_{A,0} = 4\pi\epsilon_0\frac{\omega B R^8}{L^4} \tag{20}$$

Even if the surface of ball A is no equipotential (see Eq.(2)), it can be treated as one with $V = 0$ for placing the image charges, because by the superposition principle, charges in ball A rearrange until they compensate for the tangential component of the field produced by $\boldsymbol{p}_B$ at the surface of ball A. Else, they would continue rearranging until they do so.

Ball B is placed at $x = L \gg R$, so for points close to or inside Ball A, $\sin\theta' \approx 0$ and variations in r' of the order of R can be neglected. Therefore the field from $\boldsymbol{p}_B$ is approximately uniform around ball A. The electric field of a dipole at the origin is[5]

$$\boldsymbol{E}_{\text{dip}}(r, \theta) = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}\right] \tag{21}$$

So, close to ball A,

$$\boldsymbol{E}_{p_B} \approx -\frac{p_B}{2\pi\epsilon_0(L-r)^3}\hat{\boldsymbol{x}} = -\frac{2\omega B R^8}{L^4(L-r)^3}\hat{\boldsymbol{x}}$$

With $L - r \approx L$, $\boldsymbol{E}_{p_B} \approx -\frac{2\omega B R^8}{L^7}\hat{\boldsymbol{x}}$. In the same way as above, $\boldsymbol{p}_A$ points in the negative x-direction and is given by,

$$p_A = 4\pi\epsilon_0 R^3 E_{p_B} = 8\pi\epsilon_0\frac{\omega B R^{11}}{L^7}$$

Thus, each image dipole gets another factor of $2\left(\frac{R}{L}\right)^3$, so that the correction to p_B is $p'_B = 4\left(\frac{R}{L}\right)^6 p_B$ which can be neglected since $R \ll L$ and the image series truncates. Since $p_A \ll p_B$ and since the

[5] David J. Griffiths, *Introduction to Electrodynamics*, Fifth Edition (2023), Eq.(3.104) p.156

field produced by $\boldsymbol{p}_B$ at ball A is much smaller than $E_{r,r>R}$, the force between $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$ can be neglected compared to the force of $\boldsymbol{E}_{r,r>R}$ on $\boldsymbol{p}_B$. In the same way, this shows that it was justified not to consider the force with the image charges induced in ball A by $q_{V_{\text{Im}}}$, $q_{S_{\text{Im}}}$ and $q_{C_{\text{Im}}}$.

The interaction force between ball A and ball B is equal to the force of $\boldsymbol{E}_{r,r>R}$ on $\boldsymbol{p}_B$ (the θ-dependence of the radial field and the θ-component can be neglected). The force on a dipole is[6] $\boldsymbol{F} = \boldsymbol{F}_+ + \boldsymbol{F}_- = q(\Delta\boldsymbol{E})$ and $\Delta\boldsymbol{E} = (\boldsymbol{d}\cdot\boldsymbol{\nabla})\boldsymbol{E}$, so $\boldsymbol{F} = (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}$ or, since $\boldsymbol{p}$ is constant, $\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{p}\cdot\boldsymbol{E})$. Thus,

$$\boldsymbol{F}_B = \boldsymbol{\nabla}(\boldsymbol{p}_B\cdot\boldsymbol{E}_{r,r>R})|_{r=L,\ \theta=0} = \boldsymbol{\nabla}\left(4\pi\epsilon_0\frac{\omega BR^8}{L^4}\frac{\omega BR^5}{r^4}\right)\Big|_{r=L,\ \theta=0} = -16\pi\epsilon_0\frac{\omega^2B^2R^{13}}{L^9}\hat{\boldsymbol{x}}$$

This exactly reproduces the initial result.

8 Appendix: Motivation for Approximations

8.1 Neglecting the Convection Current due to ρ and σ in Ball A

The current density due to the rotating volume charge is $\boldsymbol{J}_\rho = -2\epsilon_0\omega B\boldsymbol{\omega}\times\boldsymbol{r}$, so $J_\rho = -2\epsilon_0\omega^2Br\sin\theta$. The current density due to $\boldsymbol{B}$ and the rotation alone (which is cancelled by the current density due to the electric field) is $\boldsymbol{J}_{rotation} = \sigma_c\boldsymbol{v}\times\boldsymbol{B}$, so $J_{rotation} = \sigma_c\omega Br\sin\theta$. Comparing both gives $\frac{J_\rho}{J_{rotation}} = -\frac{2\epsilon_0\omega}{\sigma_c}$. For most metals, $\sigma_c \approx 10^7\frac{1}{\Omega\text{m}}$. So, $\frac{J_\rho}{J_{rotation}} \approx -\omega\cdot 10^{-18}\text{s}$. Thus the volume charge current as well as the surface charge current, which is of the same order of magnitude, can be neglected since ω would have to be on the order of $10^{18}\frac{1}{\text{s}}$ for the ratio to be 1. Since $R \ll \sqrt{\frac{\rho_c}{\mu_0\omega}}$, R would have to be much smaller than 10^{-10}m - so much smaller than an atom. Then we can't talk about a metal ball anymore...

The magnetic field[7] inside a spherical shell of uniform surface charge density σ is $\frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}$. While σ isn't uniform in this problem, the expression is sufficient to consider orders of magnitude. Furthermore, ρ can be divided into spherical shells, so that for both densities, taking $\sigma = \epsilon_0\omega BR$ gives a good estimation. So the magnetic field $B_{\rho,\sigma}$ created by the rotating charge densities is of the order of $(\frac{\omega R}{c})^2B$ (c = speed of light). Since $R \ll \sqrt{\frac{\rho_c}{\mu_0\omega}}$, $\omega R^2 \ll \frac{1}{\sigma_c\mu_0}$ and $B_{\rho,\sigma} \ll \frac{\omega}{\sigma_c\mu_0c^2}B = \frac{\epsilon_0\omega}{\sigma_c}B$. As shown above, the factor $\frac{\epsilon_0\omega}{\sigma_c}$ alone is already negligibly small. The change in current density J' created by $B_{\rho,\sigma}$ is much smaller than J_ρ, which is already negligible on its own: J' is of the order of $(\frac{\omega R}{c})^2B\cdot\sigma_c\omega R$ and dividing it by the order of J_ρ (which is $\epsilon_0\omega^2BR$) gives $\left(\frac{R}{\sqrt{\frac{\rho_c}{\mu_0\omega}}}\right)^2$ which is negligible by assumption.

8.2 Influence of p_B on the Mechanism creating $E_{r,r>R}$

First of all, the surface charge density σ_{p_B} induced by $\boldsymbol{p}_B$ on ball A can be compared to σ produced on its own, Eq.(9). The field created by $\boldsymbol{p}_A$ is given by Eq.(21):

$$\boldsymbol{E}_{p_A}(r,\theta) = \frac{p_A}{4\pi\epsilon_0r^3}\left[-3\cos\theta\hat{\boldsymbol{r}} + \hat{\boldsymbol{x}}\right] = -\frac{3p_A\cos\theta}{4\pi\epsilon_0r^3}\hat{\boldsymbol{r}} + \frac{2\omega BR^{11}}{L^7r^3}\hat{\boldsymbol{x}} \tag{22}$$

[6]David J. Griffiths, *Introduction to Electrodynamics*, Fifth Edition (2023), Eq.(4.5) p.170

[7]David J. Griffiths, *Introduction to Electrodynamics*, Fifth Edition (2023), Eq.(5.70) p.246; I'm not going to derive this result, since it takes quite a lot of algebra while it isn't necessary to justify that the effect of the moving charge densities can be neglected. This is already shown with the currents and the considerations with the magnetic field created by those currents are only meant as additional insight.

The total field at $r = R$ due to $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$ is then the sum of $\boldsymbol{E}_{p_A}$ as given by Eq.(22) with $r = R$ and of $\boldsymbol{E}_{p_B} \approx -\frac{2\omega BR^8}{L^7}\hat{\boldsymbol{x}}$: $\boldsymbol{E}_{p_A,p_B} = -6\frac{\omega BR^8}{L^7}\cos\theta\hat{\boldsymbol{r}}$. It follows that the induced surface charge density is

$$\sigma_{p_A,p_B} = -6\epsilon_0\frac{\omega BR^8}{L^7}\cos\theta \tag{23}$$

Comparing σ_{p_A,p_B} to Eq.(9), reveals that the effect of the dipoles can be neglected:

$$\frac{\sigma_{p_A,p_B}}{\sigma} = \frac{-6\epsilon_0\frac{\omega BR^8}{L^7}\cos\theta}{\epsilon_0\omega BR\left(\frac{5}{2}\sin^2\theta - 1\right)} \approx (\frac{R}{L})^7 \ll 1 \text{ for most } \theta \tag{24}$$

Since $\sigma_{p_A,p_B} \ll \sigma$ the current due to its rotation can be neglected as well. Furthermore, since ball A rotates in the direction in which the dipole is induced, no charges have to flow in order to keep cancelling $\boldsymbol{E}_{p_B}$ inside the ball. So there will be no additional current.

Even if $\boldsymbol{E}_{p_B}$ were not expelled from the interior of the ball by σ_{p_A,p_B}, it would have no effect since compared to the inner electric field Eq.(3), it is negligible:

$$\frac{E_{p_B}}{E_{\text{in}}} \approx (\frac{R}{L})^7 \ll 1$$