# Physics Cup 2025 P2

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## 1 Problem Statement

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Initially, four birds, labelled $A$, $B$, $C$, and $D$ are positioned at the vertices of a regular tetrahedron with side length $a$. At time $t = 0$, all birds begin flying. For all $t > 0$, each bird flies directly toward another bird in the following pattern: $A$ flies toward $B$, $B$ flies toward $C$, $C$ flies toward $D$, and $D$ flies toward $A$. For any $t > 0$, the speeds of all the birds are equal.

Calculate the total distance travelled by bird $A$ from the start until all birds meet at a single point.

**The answer is approximately** $0.8116 \cdot a$

## 2 A brief discussion

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### Symmetries

The trickiest subtlety of the problem concerns the symmetry of the trajectory. A misguided student might attempt to overuse symmetry and assume that the birds are always at the vertices of some regular tetrahedron. This approach is completely wrong: **the polyhedron formed by the birds is not a regular tetrahedron for all times** $t > 0$.

The symmetry of this problem is, in fact, slightly weaker. **There is a pairwise symmetry of opposite birds**— namely, $(A, C)$ and $(B, D)$. Based on this, we promptly conclude that, for all $t > 0$:

$$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DA} = p$$

$$\overline{AC} = \overline{BD} = q$$

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where $p$ and $q$ are time-dependent variables that characterize the setup.

## Analysing the time

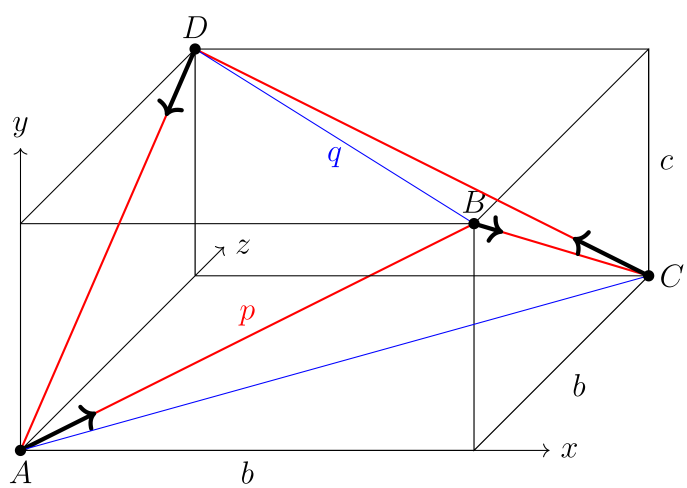
One key assertion to enable this solution is: **the trajectories do not depend on the speed of the birds**. As a matter of fact, the only relevant detail is that the speeds are the same for all birds at any given time. This allows us to assume WLOG that all birds move with a **constant velocity** $v$. Then, the problem comes down to **calculating the total time of the trajectory**.

# 3 Modelling

Now that we have successfully determined what is the shape of the tetrahedron formed by the birds, we shall mathematically describe its shape and derive differential equations that determine it.

## The cuboid representation

We have already obtained that the polyhedron $ABCD$ has 4 edges with the same length (drawn in red) and 2 other with another equal length (drawn in blue). Evidently, this “crown-like” **shape is uniquely determined by the two edge lengths $p$ and $q$**. Based on this, we may represent it as the **vertices of a square rectangular prism**, i.e. a polyhedron with two congruent opposite square faces and four congruent rectangular faces. Taking $b = \frac{q}{\sqrt{2}}$ to be the side of the square base and $c = \sqrt{p^2 - b^2}$ to be height of the prism, we obtain the desired result. This representation of the birds will be henceforth called the **cuboid representation**. This step of the solution is better understood with an image:




The mathematical proof of the existence of such a representation relies on the construction of $b$ and $c$, that satisfy the conditions. However, there still is a non-trivial detail left to prove, namely that $c$ is indeed real. In order to demonstrate that, rotate $C$ around $\overline{BD}$ such that $C'$ belongs to the $ABD$ plane. By the triangular inequality, $q = \overline{AC} \leq \overline{AC'}$, and, using Ptolemy's inequality: $2p^2 = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} = \overline{AB} \cdot \overline{C'D} + \overline{BC'} \cdot \overline{AD} \geq \overline{AC'} \cdot \overline{BD} \geq \overline{AC} \cdot \overline{BD} = q^2 \Rightarrow p^2 - b^2 \geq 0 \Rightarrow c \in \mathbb{R}$.

## Deriving some equations

Establishing the $xyz$ axis on the cuboid representation (see figure), we can analyse how the two variables that characterise the shape, $b$ and $c$, vary through time. Note that during the motion the axes are not fixed, but we set them for a given instant to study the evolution of the trajectory.

After a time $\mathrm{d}t$ has passed (i.e. the birds have flown a distance $v\mathrm{d}t$), the new positions of $A$, $B$, $C$ and $D$ are:

- $A = (0,0,0) \to A' = (v\mathrm{d}t\cos\theta, v\mathrm{d}t\sin\theta, 0)$
- $B = (b,c,0) \to B' = (b, c - v\mathrm{d}t\sin\theta, v\mathrm{d}t\cos\theta)$
- $C = (b,0,b) \to C' = (b - v\mathrm{d}t\cos\theta, v\mathrm{d}t\sin\theta, b)$
- $D = (0,c,b) \to D' = (0, c - v\mathrm{d}t\sin\theta, b - v\mathrm{d}t\cos\theta)$

where $\theta$ is the angle the red lines form with the axes, i.e., $\tan\theta = \frac{c}{b}$. Therefore, $b$ and $c$ change as follows:

$$(q + \mathrm{d}q)^2 = 2(b + \mathrm{d}b)^2 = (b - 2v\mathrm{d}t\cos\theta)^2 + (v\mathrm{d}t\sin\theta - v\mathrm{d}t\sin\theta)^2 + (b)^2$$
$$(p + \mathrm{d}p)^2 = (c + \mathrm{d}c)^2 + (b + \mathrm{d}b)^2 = (b - v\mathrm{d}t\cos\theta)^2 + (v\mathrm{d}t\cos\theta)^2 + (c - 2v\mathrm{d}t\sin\theta)^2$$

Expanding the equations and ignoring second order terms renders:

$$4b\mathrm{d}b = -4bv\mathrm{d}t\cos\theta = -4\frac{b^2v\mathrm{d}t}{\sqrt{b^2+c^2}}$$
$$2b\mathrm{d}b + 2c\mathrm{d}c = -2bv\mathrm{d}t\cos\theta - 2cv\mathrm{d}t\sin\theta = -2\frac{b^2v\mathrm{d}t}{\sqrt{b^2+c^2}} - 4\frac{c^2v\mathrm{d}t}{\sqrt{b^2+c^2}}$$

Simplifying the first and plugging into the second:

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\frac{bv}{\sqrt{b^2+c^2}} \quad (1)$$
$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{2cv}{\sqrt{b^2+c^2}} \quad (2)$$

# 4 Computation

## The invariant

From 1 and 2, it follows that:

$$2c\frac{\mathrm{d}b}{\mathrm{d}t} - b\frac{\mathrm{d}c}{\mathrm{d}t} = 0 \iff 2\frac{\mathrm{d}b}{b} - \frac{\mathrm{d}c}{c} = 0$$

Initially, $b_0 = c_0 = \frac{a}{\sqrt{2}}$.Then, integrating from the beginning to any other situation:

$$\int 2\frac{\mathrm{d}b}{b} = \int \frac{\mathrm{d}c}{c} \iff 2\ln\frac{b}{b_0} = \ln\frac{c}{c_0} \iff c = c_0\frac{b^2}{b_0^2} = \frac{b^2}{b_0} \tag{3}$$

Now, using the result from 3 on 1:

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\frac{bv}{\sqrt{b^2+c^2}} = -\frac{bv}{\sqrt{b^2+\left(\frac{b^2}{b_0}\right)^2}} = -\frac{v}{\sqrt{1+\frac{b^2}{b_0^2}}} \tag{4}$$

## The Last Integral

As discussed before, the problem comes down to calculating the total time $t_f$ from $t = 0$ until the birds meet on $t = t_f$. In this last step, we will separate and integrate 4 from the initial situation, $t = 0$ and $b = \frac{a}{\sqrt{2}}$, until the end $t = t_f$ and $b = 0$.

$$\int_{b_0}^{0} \sqrt{1+\left(\frac{b}{b_0}\right)^2}\ \mathrm{d}b = \int_0^{t_f} -v\ \mathrm{d}t$$

Substituting $b = b_0 \sinh x$ and $\mathrm{d}b = b_0 \cosh x\ \mathrm{d}x$:

$$b_0 \int_0^{\sinh^{-1}(1)} \cosh^2 x\ \mathrm{d}x = vt_f \iff$$

$$b_0 \int_0^{\sinh^{-1}(1)} \frac{\cosh 2x + 1}{2}\ \mathrm{d}x = vt_f \iff$$

$$b_0\left(\frac{\sinh\left(2\sinh^{-1}(1)\right)}{4} + \frac{\sinh^{-1}(1)}{2}\right) = vt_f \iff$$

$$\frac{a}{2\sqrt{2}}\left(\sqrt{2} + \sinh^{-1}(1)\right) = vt_f$$

As the total distance travelled is $vt_f$, the final answer is:

$$vt_f = \frac{a}{2\sqrt{2}}\left(\sqrt{2} + \sinh^{-1}(1)\right) \approx 0.8116 \cdot a$$

**Remark.** After solving the problem, I ran a C++ simulation in order to verify the correctness of my answer. The code used a $10^5$ resolution and gave a value within 0.005% of the one I obtained! :)