Physics Cup Problem 2

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Symmetries

Let us first identify the symmetries present in the original setup. An important thing to note is that there is **no tetrahedral symmetry**, as the directions of the velocities do not obey the same tetrahedral symmetry. It is convenient to illustrate the vertices as part of a unit cube. In this case, $a = \sqrt{2}$ units, and the center of the cube is at the origin O. All symmetries we find must preserve the positions of the birds as well as the direction of the velocities.

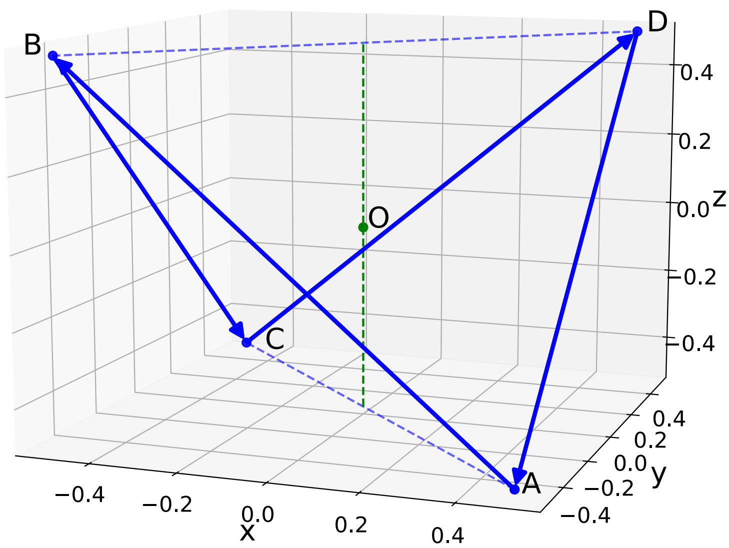
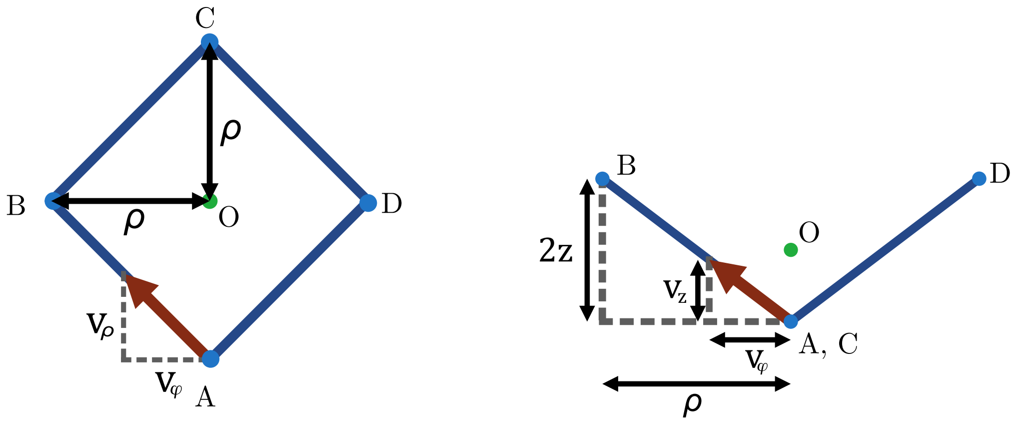


Figure 1: Initial positions and velocities of birds ($t = 0$)

There are two important symmetries. There is a 180 degree rotational symmetry about the z-axis. There is also a symmetry consisting of a reflection through the xy plane followed by a 90 degree rotation about the z-axis. These symmetries must be observed for all $t > 0$.

The former symmetry implies that A and C must have the same z coordinate and so are B and D. The latter symmetry implies that if we "squash" the birds into the xy plane (equivalent to viewing orthographically from the z-axis), $ABCD$ must always appear to be a square! This is because the "squashed" shape must have 90 degree rotational symmetry.

Hence, for $t > 0$, the 4 birds are arranged in a tetragonal disphenoid. Intuitively, it is simply a regular tetrahedron like the one shown in Figure 1, but rescaled in the z-axis. Orthographic views are shown in Figure 2, where the red arrow represents the velocity of bird A and the blue lines are the lines of the birds' velocities. We define z_i as the magnitude of the z-coordinate of bird i and ρ_i as its radial distance in cylindrical coordinates as measured from the origin. They are functions of time, and due to the aforementioned symmetries, $z_A = z_B = z_C = z_D = z$ and $\rho_A = \rho_B = \rho_C = \rho_D = \rho$.



(a) View from +z axis

(b) View from CA produced

Figure 2: Orthographic views for $t > 0$

Trajectories

The function $v(t)$ doesn't change the spatial path of the birds at all, so we can just choose the simplest nonzero function, by assuming that the speed of the birds are constant for all $t > 0$. Letting v be the speed of each bird, we express the speeds in terms of cylindrical coordinates. Note that $v_\phi = \mathbf{v} \cdot \hat{\phi}$, etc.

$$v_z^2 + v_\phi^2 + v_\rho^2 = v^2 \tag{1}$$

$ABCD$ forms a square in Figure 2a, so $v_\phi = v_\rho$. Then, using similar triangles on Figure 2b,

$$\frac{v_\rho}{v_z} = \frac{v_\phi}{v_z} = \frac{\rho}{2z}$$

Substituting the above results into Equation 1,

$$\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{v}{\sqrt{1 + \rho^2/2z^2}} \tag{2}$$

$$\dot{\rho} = -\frac{\rho/2z}{\sqrt{1 + \rho^2/2z^2}} v \tag{3}$$

Dividing equation 2 by equation 3 and using the chain rule,

$$\frac{\dot{z}}{\dot{\rho}} = \frac{\mathrm{d}z}{\mathrm{d}\rho} = \frac{2z}{\rho}$$

Separating and integrating,

$$\frac{z}{z_0} = \left(\frac{\rho}{\rho_0}\right)^2 \tag{4}$$

where z_0 and ρ_0 are the original values of z and ρ respectively. Doing some geometry on Figure 1, $z_0 = a/\sqrt{8}$ and $\rho_0 = a/2$. Substituting equation 4 into equation 2,

$$\dot{z} = -\frac{v}{\sqrt{1 + a/z\sqrt{8}}}$$

Separating by variables and applying $d = vt$,

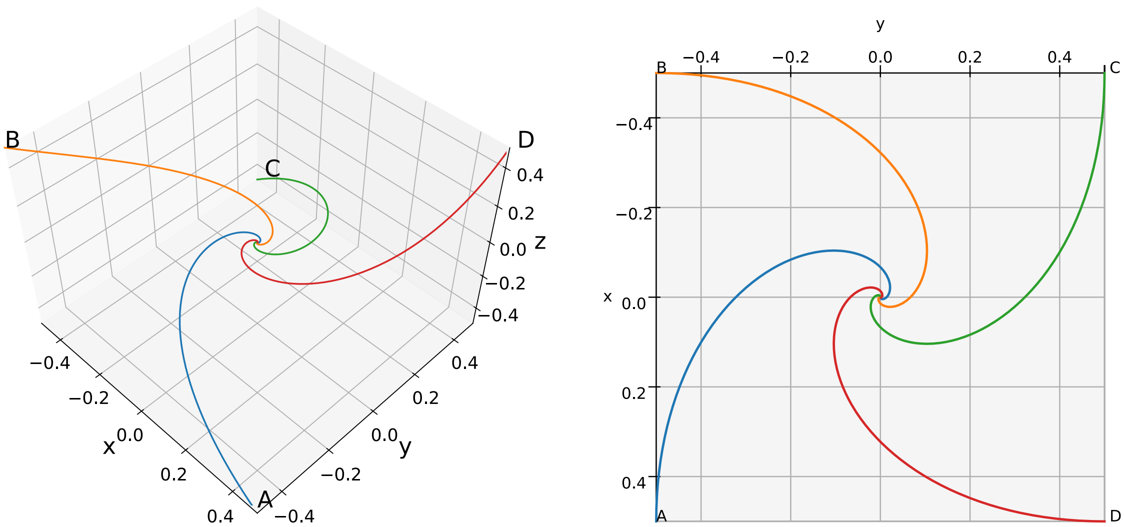
$$d = \int_0^{z_0} \sqrt{1 + \frac{a}{z\sqrt{8}}} dz = a \int_0^{1/\sqrt{8}} \sqrt{1 + \frac{1}{u\sqrt{8}}} du$$

where we nondimensionalised the integral in the last step by using the substitution $u \equiv z/a$. Evaluating the integral is extremely messy, but we obtain

$$\boxed{d = \left(\frac{1}{4\sqrt{2}} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) + \frac{1}{2} \right) a \approx 0.812a} \tag{5}$$

Simulation

This problem is fairly simple to simulate, and it confirms our calculations. We can also look at the plot from above (Figure 3b) and confirm that there is 90 degree rotational symmetry. In fact, the shape in figure 3b is known as a logarithmic spiral, and is also observed in a similar 2-dimensional setup where the birds are arranged in a square.



(a) Perspective view

(b) Orthographic view from +z axis

Figure 3: Simulation