Physics Cup Problem 2

Tetrahedral flock of birds

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1 Introduction

This solution begins with a qualitative description of the problem, showing how it can be reduced to a classical case by projecting the velocities. The discussion then transitions to the quantitative aspects of the solution.

2 Qualitative description

2.1 Initial tetrahedron

A plane is defined that passes through the line BD and is parallel to the side AC. Within this plane, the points A and C are projected onto A' and C', respectively. Both BD and $A'C'$ have a length of a, while the projections of the other sides are equal to x_0 due to symmetry.

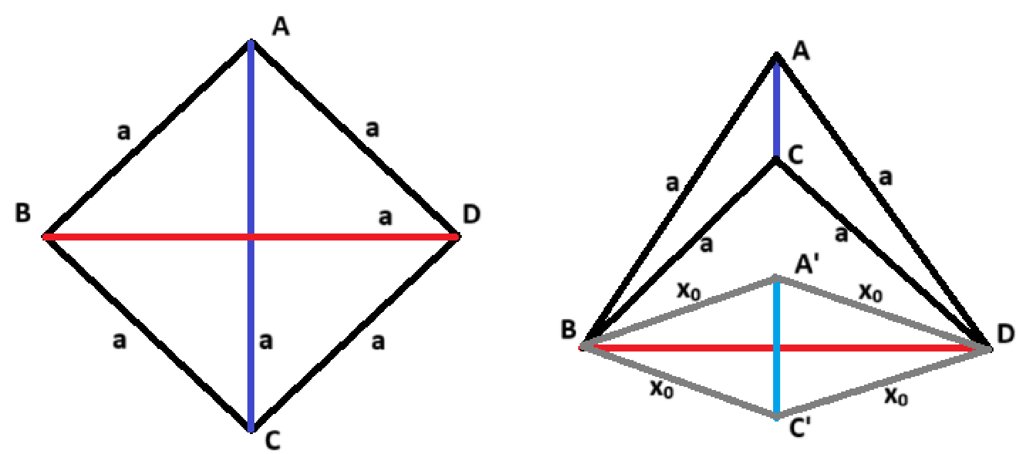

Figure 1: Top view in 2D and side view of the tetrahedron

The horizontal projections of the points form a square with side length x_0. Points A and C are at a height h_0 over the plane, while B and D lie at height zero on the plane.

2.2 Horizontal movement analysis

The angles between the sides AB, BC, CD and DA; and their projections x are equal, and their tangent satisfies:

$$\tan(\alpha) = \frac{h}{x}.$$

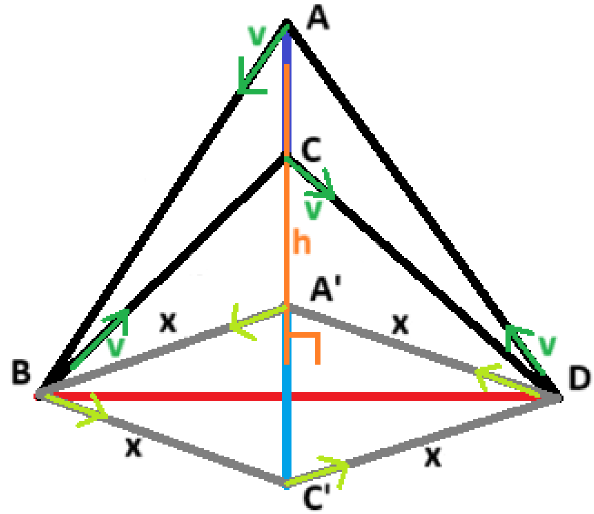

Figure 2: Illustration of bird movements

On the horizontal plane, all points follow one another at the same horizontal velocity, given by $v\cos(\alpha) = v\sin\left(\frac{\pi}{2} - \alpha\right)$.
Initially, points B, D, A' and C' form a square of side x_0. This configuration reduces the problem to the classical 2D following problem on a square, where the side length x decreases over time but the square shape remains intact.

2.3 Vertical movement analysis

In the previous subsection, we established that the horizontal velocity of each point is $v\cos(\alpha)$. Referring to Figure 2, we observe: - Points A and C move downward with velocity $v\sin(\alpha)$, - Points B and D move upward with the same velocity.

3 Quantitative solution

3.1 Geometric determination of initial values

The initial height h_0 can be determined using the triangle DCA and the Pythagorean Theorem:

$$h_0^2 + \left(\frac{a}{2}\right)^2 = \left(\frac{\sqrt{3}a}{2}\right)^2,$$

$$h_0 = \frac{a}{\sqrt{2}}.$$

The initial horizontal projection x_0 can also be found using the Pythagorean Theorem:

$$x_0^2 + h_0^2 = a^2,$$

$$x_0 = \frac{a}{\sqrt{2}}.$$

3.2 System of differential equations

Since (as stated in Section 2.3) both BD and AC reduce h moving vertically, the equations for x and h are:

$$dx = -v\cos(\alpha)\,dt,$$

$$dh = -2v\sin(\alpha)\,dt.$$

By dividing these equations, we find the relationship between h and x:

$$\frac{dh}{dx} = 2\tan(\alpha) = 2\frac{h}{x}.$$

Integrating both sides:

$$\int_{h_0}^{h} \frac{dh}{h} = 2\int_{x_0}^{x} \frac{dx}{x},$$

$$\frac{h}{h_0} = \frac{x^2}{x_0^2},$$

$$h = \frac{\sqrt{2}x^2}{a}.$$

Therefore when $x = 0$, h is also 0. From the relationship $\tan(\alpha) = \frac{h}{x} = \frac{\sqrt{2}x}{a}$, we can derive:

$$\cos(\alpha) = \frac{1}{\sqrt{1 + \frac{2x^2}{a^2}}}.$$

To calculate the time required for all points to meet, we find the time at which the side length of the square becomes zero:

$$\int_{x_0}^{0} \sqrt{1 + \frac{2x^2}{a^2}}\, dx = \int_0^t -v\, dt.$$

Evaluating this integral:

$$\frac{a}{4}\left(2 + \sqrt{2}\,\text{arcsinh}(1)\right) = vt.$$

Finally, multiplying v by the total time gives the total distance d traveled by a point. Thus, the result is:

$$d = \frac{a}{4}\left(2 + \sqrt{2}\,\text{arcsinh}(1)\right) \approx 0.81a.$$