# Problem 2 - Solution

It might seem that the birds always form a scaled and rotated regular tetrahedron because of symmetry, but this is not the case. To correctly solve the problem, we must carefully look at the separation vector between any 2 birds, say $A$ and $B$. Notice that, in the beginning, the birds can be placed onto every mutually non-neighbouring vertex of a cube as they form a regular tetrahedron. This motivates us to introduce a cylindrical coordinate system whose origin, denoted $O$, is the cube's centroid ($z$-axis pointing upwards):

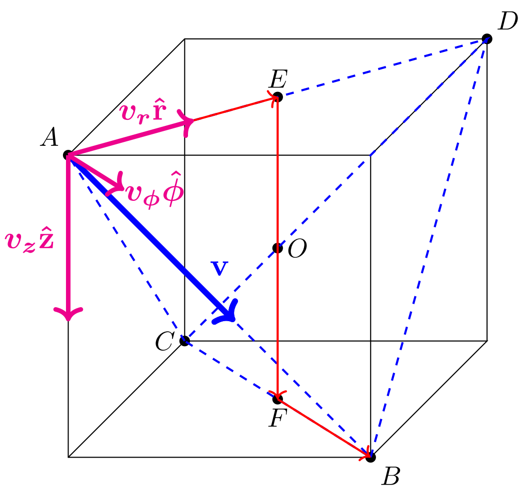


Figure 1: The system at $t = 0$

This system is characterised by 90°-rotational axial symmetry for the radial and azimuthal coordinates, $r$ and $\phi$, and 90°-rotational axial antisymmetry for the axial coordinate, $z$, whose sign is alternating between two neighbouring birds. Now, we express the separation vector between the birds $A$ and $B$ in cylindrical coordinates as $\mathbf{l} = \mathbf{x}_\mathrm{B} - \mathbf{x}_\mathrm{A}$. We trace the path from $A$ to $B$ along the radial, axial and azimuthal directions, hence now we use the points $E$ and $F$ as drawn in figure 1:

$$\mathbf{l} = (\mathbf{x}_\mathrm{B} - \mathbf{x}_\mathrm{F}) + (\mathbf{x}_\mathrm{F} - \mathbf{x}_\mathrm{E}) + (\mathbf{x}_\mathrm{E} - \mathbf{x}_\mathrm{A})$$

We notice the segment $\overline{AE}$, whose length is $r$, being parallel to $\hat{\mathbf{r}}$ at $A$, $\overline{EF}$, whose length is $2z$, being parallel to $\hat{\mathbf{z}}$, and $\overline{BF}$, whose length is $r$ as well, being parallel to $\hat{\boldsymbol{\phi}}$ at $A$. We rewrite the separation vector:

$$\mathbf{l} = (r\hat{\boldsymbol{\phi}}) + (-2z\hat{\mathbf{z}}) + (-r\hat{\mathbf{r}})$$

We may now use the fact that each bird's velocity has the same direction as the separation vector between itself and the bird it's following, as described in the problem:

$$\mathbf{v} = v\frac{\mathbf{l}}{l} = v_r\hat{\mathbf{r}} + v_z\hat{\mathbf{z}} + v_\phi\hat{\boldsymbol{\phi}}$$

We can now easily identify each of the velocity components in the cylindrical coordinate system chosen, therefore we write for each component:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v_r = -\frac{vr}{l}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z = -\frac{2vz}{l}$$

$$r\frac{\mathrm{d}\phi}{\mathrm{d}t} = v_\phi = \frac{vr}{l}$$

The total distance differential is given by essentially applying Pythagoras' theorem to the differentials of distances travelled in each direction of our coordinate system:

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + (r\mathrm{d}\phi)^2 + \mathrm{d}z^2} = \mathrm{d}r\sqrt{1 + \left(\frac{r\mathrm{d}\phi}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}$$

From the derivative chain rule, we can easily evaluate the squared terms:

$$r\frac{\mathrm{d}\phi}{\mathrm{d}r} = r\frac{\mathrm{d}\phi}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^{-1} = \frac{vr}{l}\left(-\frac{l}{vr}\right) = -1$$

It will only be necessary to solve the differential equation for $z(r)$:

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\mathrm{d}z}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^{-1} = -\frac{2vz}{l}\left(-\frac{l}{vr}\right) = \frac{2z}{r}$$

$$\frac{\mathrm{d}z}{z} = \frac{2\mathrm{d}r}{r} \implies \ln(z) = 2\ln(r) + C$$

We rewrite the last equation as $z(r) = Ar^2$. For $t = 0$, $r$ is half the tetrahedron's side length, while $z$ is simply given by Pythagoras' theorem on the isosceles right triangle, hence:

$$z\left(\frac{a}{2}\right) = \frac{a\sqrt{2}}{4} \implies z(r) = \frac{r^2\sqrt{2}}{a}$$

Now, we simply plug the obtained squared terms back into the distance differential:

$$\mathrm{d}s = \mathrm{d}r\sqrt{1 + (-1)^2 + \left(\frac{2r\sqrt{2}}{a}\right)^2} = \mathrm{d}r\sqrt{2 + 2\left(\frac{2r}{a}\right)^2}$$

We integrate this differential for the total distance travelled by each bird. The appropriate boundaries are $r = a/2$ at $t = 0$, and $r = 0$ when all birds meet up. Afterwards, we substitute $r = ax/2$, giving us a non-dimensional integral:

$$s = \int_0^{a/2}\sqrt{2\left(1 + \left(\frac{2r}{a}\right)^2\right)}\,\mathrm{d}r = \frac{a\sqrt{2}}{2}\int_0^1\sqrt{1 + x^2}\,\mathrm{d}x$$

The integral is easily evaluated after performing hyperbolic substitution:

$$\int\sqrt{1 + x^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\ln\left(x + \sqrt{1 + x^2}\right) + C$$

We conclude with our final answer to the problem:

$$s = \frac{a}{2}\left(1 + \frac{\ln(1 + \sqrt{2})}{\sqrt{2}}\right) \approx 0.812a$$