PC 2025 task 2

Karlo Maksimović

18.11.2024.

1 Starting observations

We may notice that our starting system, being a tetrahedron is extremely symmetric. Specifically, we may notice that the relations between A and B/D are the same, analogously this holds true for the relations between B and A/C, C and B/D, D and A/C. Keeping this in mind will help us choose a preferable way to work with a tetrahedron and help us find useful symmetries.
Since the speed of the birds is constant, we may consider it to be v and say that our task is finding the time t required for the birds to meet, so the total path will simply be $s = vt$.

2 The Carthesian vertices of a tetrahedron

There are many ways to represent a tetrahedron in Carthesian coordinates, since the system is symmetric and will obviously tend towards the center point of the tetrahedron, we shall pick coordinates where the center of the tetrahedron is the origin of our system.
We shall use the representation where the vertices of a tetrahedron are vertices of a cube with side length $\frac{a}{\sqrt{2}}$:

$$A = \left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right) \tag{1}$$

$$B = \left(-\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}, -\frac{a}{2\sqrt{2}}\right) \tag{2}$$

$$C = \left(-\frac{a}{2\sqrt{2}}, -\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right) \tag{3}$$

$$D = \left(\frac{a}{2\sqrt{2}}, -\frac{a}{2\sqrt{2}}, -\frac{a}{2\sqrt{2}}\right) \tag{4}$$

Here we may notice that a cylindrical system may be preferable, due to the radial distance being constant among all the birds and due to the angle formed with the x-axis not mattering when trying to determine the time required for the birds to meet (which will obviously happen at the origin due to any other point causing asymmetry). With this we can rewrite the coordinates in (r, ϕ, z)

as such:

$$
\begin{aligned}
A = & \quad (\tfrac{a}{2}, \tfrac{\pi}{4}, \tfrac{a}{2\sqrt{2}}) && (5)\\
B = & \quad (\tfrac{a}{2}, \tfrac{3\pi}{4}, \tfrac{a}{-2\sqrt{2}}) && (6)\\
C = & \quad (\tfrac{a}{2}, \tfrac{5\pi}{4}, \tfrac{a}{2\sqrt{2}}) && (7)\\
D = & \quad (\tfrac{a}{2}, \tfrac{7\pi}{4}, \tfrac{a}{-2\sqrt{2}}) && (8)
\end{aligned}
$$

To get the direction of the velocity we may notice that the velocity in a particular direction is proportional to the difference between the 2 neighboring birds.
First we can notice that the angle difference between 2 neighboring birds in $\frac{\pi}{4}$ and that their radii are the same. From that we can infer that the radial and angular velocity are of the same magnitude.
Furthermore, the z-axis distance between any 2 neighboring birds is the same, so in other words each bird will have the same magnitude for all 3 velocities (radial, angular and along the z-axis), with only the direction of the z-axis velocity alternating between birds.
Because of that, it's simple to see that the difference between angles will stay $\frac{\pi}{4}$ and if we label the radial distance at a given time of the birds from the origin as $r(t)$ and their z coordinate as $\frac{z(t)}{2}$ (with $z(t)$ being their height difference, $r > 0, z > 0$), we can easily get:

$$
\begin{aligned}
|v_r(t)| = & \quad |v_\phi(t)| = |v_z(t)|\tfrac{r(t)}{z(t)} = v\tfrac{r}{\sqrt{z^2+2r^2}} && (9)\\
|v_z(t)| = & \quad v\tfrac{z}{\sqrt{z^2+2r^2}} && (10)\\
r(0) = & \quad \tfrac{a}{2} && (11)\\
z(0) = & \quad \tfrac{a}{\sqrt{2}} && (12)
\end{aligned}
$$

Since we only care about the radial and z-axis distances, we shall now label the coordinates as (r, z). Furthermore, due to the symmetry of the system it is adequate to look at only one bird. Looking at what happens in a short time dt:

$$
dA = (dr, \frac{dz}{2}) = (-|v_r(t)|dt, -|v_z(t)|dt) = (-vdt\frac{r}{\sqrt{z^2+2r^2}}, -vdt\frac{z}{\sqrt{z^2+2r^2}}) \tag{13}
$$

From this we can get:

$$
\begin{aligned}
\frac{dr}{dz} &= \frac{r}{2z} && (14)\\
z &= Cr^2 && (15)\\
z(0) &= Cr^2(0) && (16)\\
\frac{a}{\sqrt{2}} &= C\frac{a^2}{4} && (17)\\
C &= \frac{\sqrt{8}}{a} && (18)
\end{aligned}
$$

$$dr = -vdt\frac{r}{\sqrt{8\frac{r^4}{a^2} + 2r^2}} \tag{19}$$

$$\int_{\frac{a}{2}}^{0} \sqrt{8\frac{r^2}{a^2} + 2}dr = \int_{0}^{t} -vdt \tag{20}$$

$$\sqrt{2}\int_{0}^{\frac{a}{2}} \sqrt{(\frac{2r}{a})^2 + 1}dr = vt \tag{21}$$

$$u = \frac{2r}{a}; du = dr\frac{2}{a} \tag{22}$$

$$\frac{a}{\sqrt{2}}\int_{0}^{1} \sqrt{u^2 + 1}du = vt \tag{23}$$

$$vt = \frac{a}{\sqrt{2}}\frac{1}{2}(\sqrt{2} + \ln\left(\sqrt{2} + 1\right)) \tag{24}$$

$$vt = a\frac{2 + \sqrt{2}\ln\left(\sqrt{2} + 1\right)}{4} \tag{25}$$

$$vt \approx 0.8116a \tag{26}$$

Since the birds velocity is constant, vt is simply the path the birds traveled, so the total distance is $a\frac{2+\sqrt{2}\ln\left(\sqrt{2}+1\right)}{4}$
Here is a quick sketch of their paths made in python, the source code is below. Interestingly, the projection onto the $r - \phi$ plane is radically different from the case of 4 birds in the vertices of a square in a plane, which is shown in the 3rd graph.

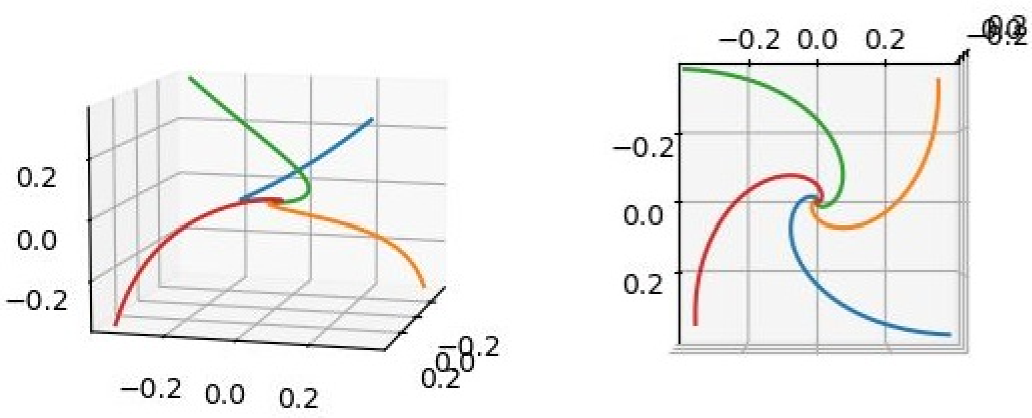

Figure 1: The path of the birds, shown from the side and from above respectively

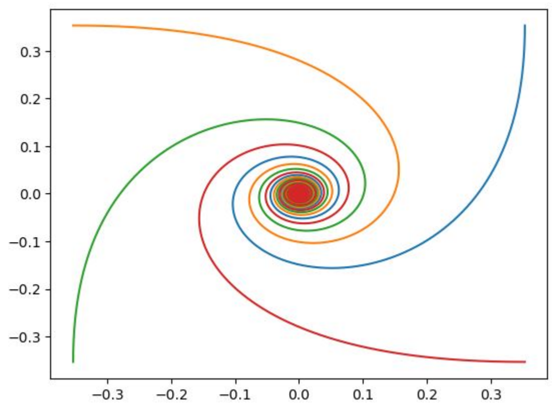

Figure 2: The path of birds in the vertices of a square in a plane

```
import numpy as np
import matplotlib.pyplot as plt
import mpl_toolkits.mplot3d.axes3d as p3
from scipy.integrate import odeint
def f1(r,t):
    return -v/np.sqrt(8*r**2/a**2+2)
a=1
v=1
r0=a/2
ts=np.linspace(0,0.8116*a/v,200)
R= odeint(f1,r0,ts)
Z=np.sqrt(8)/a*R**2
P=[np.pi/4]
for i in range(len(R)-1):
    P.append(P[i]+(R[i+1][0]-R[i][0])/R[i][0])
P=np.array(P)
fig = plt.figure()
ax1=fig.add_subplot(1,2,1,projection='3d')
ax2=fig.add_subplot(1,2,2,projection='3d')

for i in range(4):
    X, Y = R[:,0]*np.cos(P), R[:,0]*np.sin(P)
    ax1.plot3D(X, Y, Z[:,0]/2)
    ax2.plot3D(X, Y, Z[:,0]/2)
    P+=np.pi/2
    Z=Z*(-1)
ax1.view_init(elev=10,azim=15)
ax2.view_init(elev=90,azim=0)
```

And the code in the 2nd cell for the square:

```
ts=np.linspace(0,np.sqrt(2)*a/2/v-0.001,2000)
R2=a/2 - v*ts/np.sqrt(2)
P2=np.pi/4 - v/np.sqrt(2)*ts/R2
for i in range(4):
    plt.plot(R2*np.cos(P2),R2*np.sin(P2))
    P2+=np.pi/2
```