PROBLEM 2: TETRAHEDRAL FLOCK OF BIRDS

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Problem

Initially, four birds, labeled A, B, C, D, are positioned at the vertices of a regular tetrahedron with side length a. At time t=0, all birds begin flying. For all $t > 0$, each bird flies directly toward another bird in the following pattern: A flies toward B, B flies toward C, C flies toward D, and D flies toward A. For any $t > 0$ the speeds of all the birds are equal. Calculate the total distance traveled by bird A from the start until all birds meet at a single point.

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1 Solution

1.1 General consideration

- Since the speed of all birds are the same and costant, the distance traveled by bird A can be calculated as

$$s = vT \tag{1}$$

  where T represents the time difference between the start of the movement and collision.

- Due to symmetry, sides AB, BC, CD and DA will change equally and will have the same length throughout the movement. However sides BD and AC will change in different manner. Hence we will end up with the body, where 4 specific sides will have equal length and the other 2 will have same length.

1.2 Figures

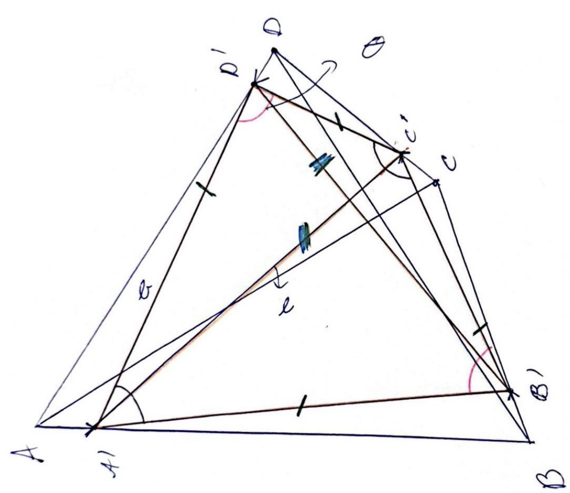

Figure 1: Sketch 1

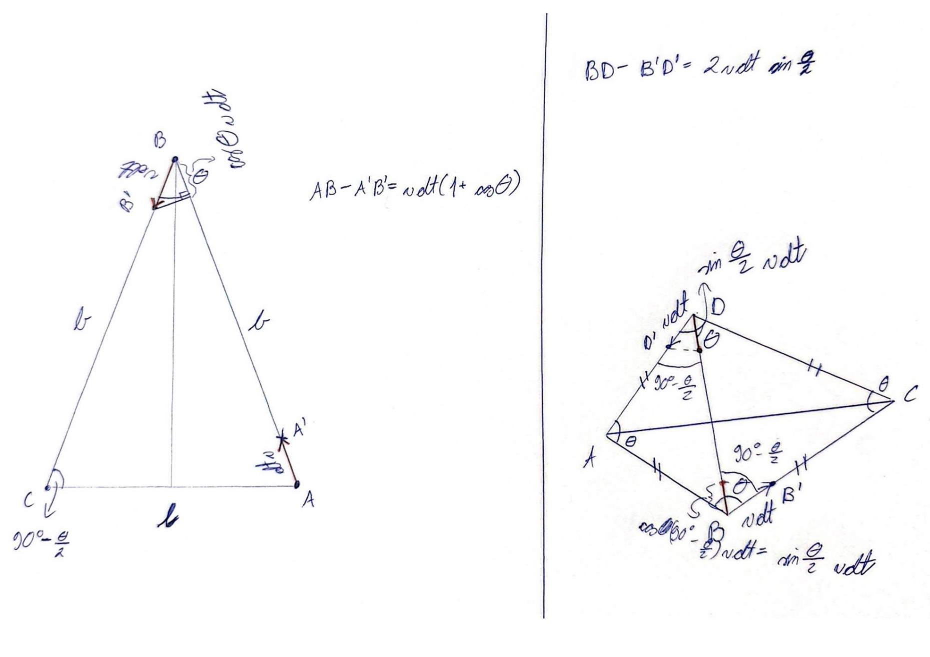

Figure 2: Sketch 2

1.3 Rate of changes

Let the length of sides AB, BC, CD, DA, be b and let the length of the sides BD and AC be l. As can be seen from the figure (2), the angle between the respective lines of length b is donated as θ. Quantities l, b and θ are connected trough the equation:

$$\sin(\frac{\theta}{2})\ b = \frac{l}{2} \tag{2}$$

As can be seen from the figure (2), the rate of change of specific side can be calculated in the following way: After time dt, each bird moves the distance $v\,dt$. After this time, 2 things happen with the side AB. Firstly it gets contracted by the length $v\ dt\ (1+\cos\theta)$ and is also rotated for an angle $\frac{\sin\theta v dt}{b}$. In my solution, the rotation angle does not play any significant role and only the derivatives of b and l with respect to time will be needed for solution. As can be seen from the figure (2), rate of changes for b and l are respectivly:

$$\frac{db}{dt} = -v(1+\cos\theta) \tag{3}$$

$$\frac{dl}{dt} = -2v\sin\frac{\theta}{2} = -2v\frac{l}{2b} = -v\frac{l}{b} \tag{4}$$

In equation (3), we need to get rid of theta, because it's derivative is unknown. This can be done with the formula for double angle:

$$1 - 2\sin^2\frac{\theta}{2} = \cos\theta$$

Combinig this with equation (3), we get:

$$\frac{l^2}{2b^2} = 2 + \frac{db}{dt}\frac{1}{v} \tag{5}$$

Combining (5) and (4) we get:

$$\frac{l^2}{2b^2} = 2 - \frac{db}{dl}\frac{l}{b} \tag{6}$$

We multiply the whole equation with $\frac{2b^2}{l^2}$ (We can do this, only when b and l are not zero. Since b and l are only zero, at the end of the process considered in the problem, we can at the end just take the limit of the final function. This will not bring any complications throughout the solution):

$$1 = 4\frac{b^2}{l^2} - 2\frac{b}{l}\frac{db}{dl} \tag{7}$$

We can now switch the variables in the following way:

$$y = b^2$$

$$dy = 2b\ db$$

$$x = l^2$$

$$dx = 2l\ dl$$

Switching to new variables:

$$1 = 4\frac{y}{x} - 2\frac{dy}{dx} \tag{8}$$

We can solve this by introducing the new variable $v = \frac{y}{x}$. Switching the variable for the second time gives:

$$1 = 4v - 2v - 2x\frac{dv}{dx} = 2v - 2x\frac{dv}{dx} \tag{9}$$

This equation can be easily solved by the separation of variables:

$$\int \frac{1}{x}\ dx = \int \frac{1}{v-\frac{1}{2}}\ dv \tag{10}$$

$$\ln\frac{\mid x \mid}{\mid x_0 \mid} = \ln\frac{\mid v-\frac{1}{2} \mid}{\mid v_0 - \frac{1}{2} \mid} \tag{11}$$

Returning to the original variables, equation (11) gives:

$$\frac{l^2}{l_0^2} = \frac{\mid \frac{b^2}{l^2} - \frac{1}{2} \mid}{\mid \frac{b_0^2}{l_0^2} - \frac{1}{2} \mid} \tag{12}$$

On left side, I have dropped the absolute sign value, since for our problem $\frac{l^2}{l_0^2} > 0$. Now when the motion starts, birds form a tetrahedron, so $b_0 = l_0 = a$:

$$\frac{1}{2}\frac{l^2}{a^2} = \mid \frac{b^2}{l^2} - \frac{1}{2} \mid \tag{13}$$

$$\frac{1}{2}\frac{l^4}{a^2} = \mid b^2 - \frac{1}{2}l^2 \mid \tag{14}$$

Graph of this equation is plotted in section (2). Looking at the graph, one can see 2 curves in the 1. quadrant (region of problem's interest). By the argument of continuity (in time dt, b can only change by a differential order of magnitude) and because of our initial conditions, we consider only the curve going trough point (a,a)(point (5,5) on graph). This curve is described by the equation:

$$\frac{1}{2}\frac{l^4}{a^2} = b^2 - \frac{1}{2}l^2 \tag{15}$$

$$\frac{1}{2}\frac{l^4}{a^2} + \frac{1}{2}l^2 = b^2 \tag{16}$$

1.4 Time

With equation (16), we can now calculate the time. Combined with equation (4), we get the equation:

$$\int_0^T dt = -\int_a^0 \frac{b}{vl}\, dl = \int_0^a \frac{1}{v\sqrt{2}}\sqrt{1+\frac{l^2}{a^2}}\, dl \tag{17}$$

$$T = \frac{a}{v\sqrt{2}}\int_0^1 \sqrt{1+x^2}\, dx$$

Integral $\int_0^1 \sqrt{1+x^2}\, dx$ represents surface under regular hyperbola. Surface under hyperbola can be thought as 2 contributions as seen from the figure (3):

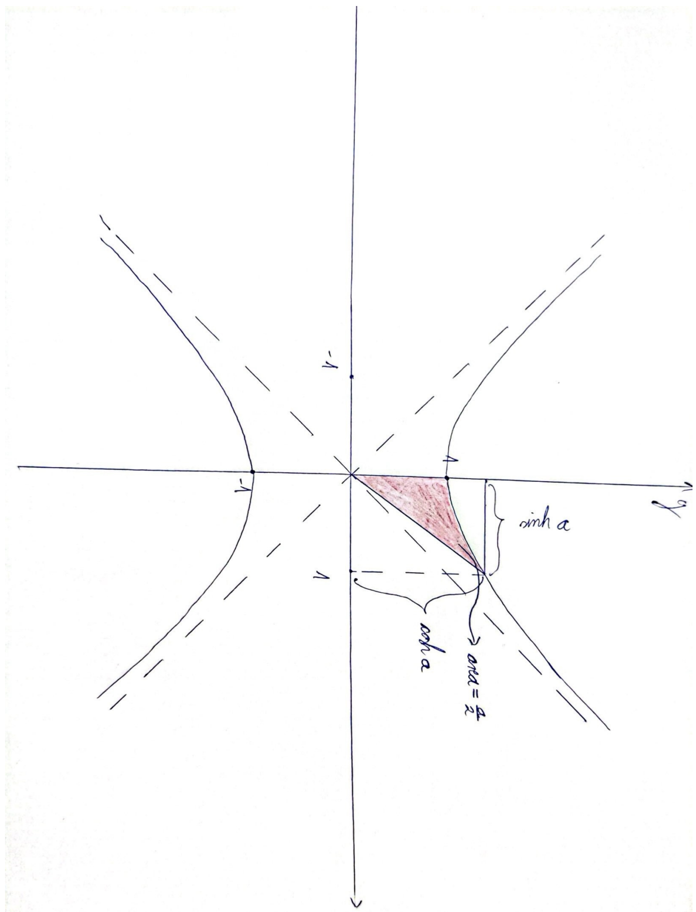

Figure 3: Hyperbola

- right triangle with catheti of length sides 1 and $\sqrt{2}$
- red surface in figure (3), with surface $\frac{1}{2}\sinh^{-1} 1$

Since $\sinh^{-1} 1 = \ln(1+\sqrt{2})$, we get out final answer:

$$T = \frac{a}{v\sqrt{2}}(\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})) \tag{18}$$

$$T = \frac{a}{2v}(1 + \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})) \tag{19}$$

And the distance traveled by the bird A is:

$$s = \frac{a}{2}(1 + \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})) \tag{20}$$

2 Grapf(a=5)

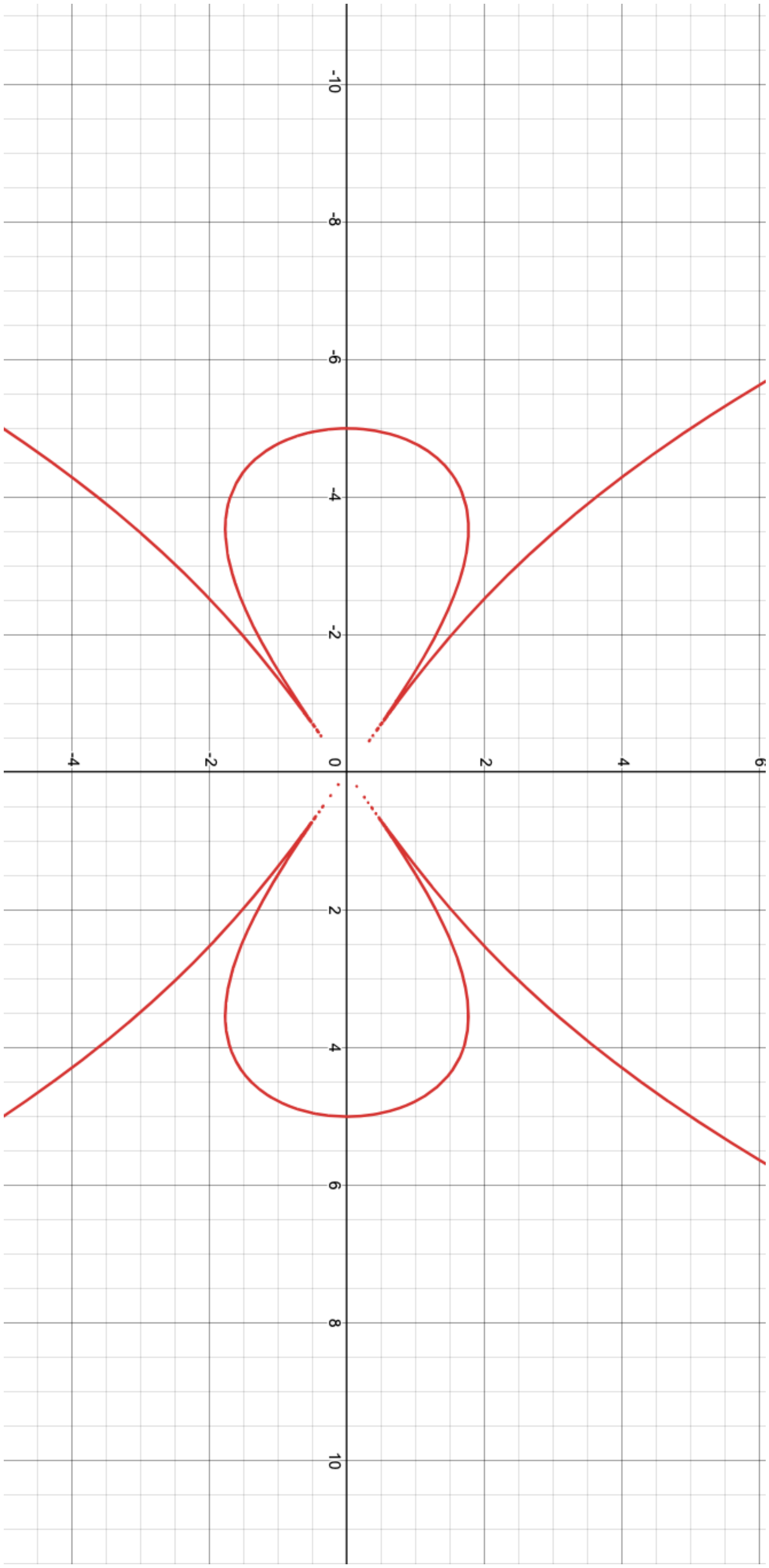

Figure 4: Graph b versus l for a=5