2025 Physics Cup Problem 2

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1 Lemmas

Lemma 1.1. *At all times* $t \geq 0$*,* $AB = BC = CD = DA$ *and* $AC = BD$*.*

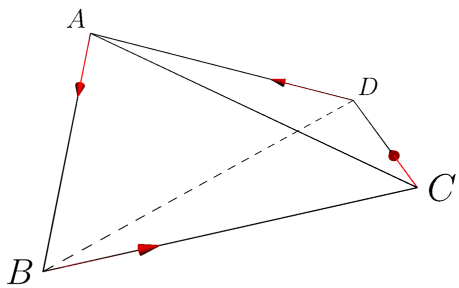
Proof. Suppose that at some point in time, $AB = BC = CD = DA$ and $AC = BD$. Since $\triangle ABC \cong \triangle BCD \cong \triangle CDA \cong \triangle DAB$, We can thus permute the labels cyclically $(A, B, C, D) \to (B, C, D, A)$, producing a situation physically indistinguishable from the original, aside from rotation. From this relabelling, we have

$$\frac{d}{dt}AB = \frac{d}{dt}BC = \frac{d}{dt}CD = \frac{d}{dt}DA$$

and

$$\frac{d}{dt}AC = \frac{d}{dt}BD.$$

Thus, the initial condition remains true. Since at the start, all pairwise distances are equal, we are done. □



Lemma 1.2. *Let* $\mathbf{u} \in \mathbb{R}^3$*. We have*

$$\frac{d}{dt}\|\mathbf{u}\| = \operatorname{comp}_{\mathbf{u}} \frac{d\mathbf{u}}{dt}.$$

Proof.

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}\| &= \frac{d}{dt}\sqrt{\mathbf{u}\cdot\mathbf{u}} = \frac{1}{2\|\mathbf{u}\|}\left(2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\right) \\ &= \frac{d\mathbf{u}}{dt}\cdot\frac{\mathbf{u}}{\|\mathbf{u}\|} = \operatorname{comp}_{\mathbf{u}}\frac{d\mathbf{u}}{dt}.\end{aligned}$$

□

Corollary 1.3. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$*. We have*

$$\frac{d}{dt}\|\mathbf{u} - \mathbf{v}\| = \operatorname{comp}_{\mathbf{u}-\mathbf{v}} \frac{d\mathbf{u}}{dt} + \operatorname{comp}_{\mathbf{v}-\mathbf{u}} \frac{d\mathbf{v}}{dt}.$$

2 Solution

Since the birds always travel at the same speeds, the final trajectory and distance traveled will be the same irrespective of the particular speeds during the flight. We can thus assume that the birds always fly with speed v. Let $AB = BC = CD = DA = x$, $AC = BD = y$, and $\angle ABC = \angle BCD = \angle CDA = \angle DAB = \theta$. Projecting the velocities of the birds as in Corollary 1.3, we get

$$\frac{dx}{dt} = -v - v\cos\theta.$$

Since $\angle CAB = \angle ACD = \angle BDA = \angle DBC = \frac{\pi}{2} - \frac{\theta}{2}$, we have

$$\frac{dy}{dt} = -2v\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = -2v\sin\frac{\theta}{2}.$$

From law of cosines and half angle identity/geometry, we get

$$\cos\theta = \frac{x^2 + x^2 - y^2}{2x^2} = 1 - \frac{y^2}{2x^2}, \quad \sin\frac{\theta}{2} = \frac{y}{2x}.$$

This allows us to rewrite the equations as

$$\frac{dx}{dt} = -v\left(2 - \frac{y^2}{2x^2}\right) \quad \text{and} \quad \frac{dy}{dt} = -v\frac{y}{x}.$$

We make the substitutions $X = x^2$ and $Y = y^2$, giving

$$\frac{dX}{dt} = 2x\frac{dx}{dt} = -2xv\left(2 - \frac{Y}{2X}\right),$$

and

$$\frac{dY}{dt} = 2y\frac{dy}{dt} = -2v\frac{y^2}{x} = -2xv\frac{Y}{X}.$$

Dividing the two, we get

$$\frac{dX}{dY} = \frac{2X}{Y} - \frac{1}{2}.$$

Observe the following solution:

$$X = cY^2 + \frac{1}{2}Y \implies x = \sqrt{cy^4 + \frac{1}{2}y^2},$$

for $c \in \mathbb{R}$. Since this is a first order differential equation, we have found all solutions to the equation. Since we have the initial condition $x = y = a$, this gives $c = \frac{1}{2a^2}$, giving

$$x = \sqrt{\frac{1}{2a^2}y^4 + \frac{1}{2}y^2}.$$

Rearranging and substituting back into the original equation, we get

$$-\frac{x}{y}dy = vdt \implies -\sqrt{\frac{1}{2a^2}y^2 + \frac{1}{2}}dy = vdt.$$

Note that vdt is the distance covered in time dt. Integrating from start to finish, we get

$$-\int_a^0 \sqrt{\frac{y^2/a^2 + 1}{2}}dy = D,$$

where D is the total distance covered. Under the substitution $\beta = \frac{y}{a}$, we get

$$D = \frac{a}{\sqrt{2}}\int_0^1 \sqrt{1 + \beta^2}d\beta.$$

Under the substitution $\beta = \tan\alpha$, we can rewrite the integral as

$$\int_0^1 \sqrt{1+\beta^2} d\beta = \int_0^{\pi/4} \sec^3\alpha \; d\alpha.$$

Integrating by parts, we get

$$\begin{aligned}\int_0^{\pi/4} \sec^3\alpha \; d\alpha &= \sec\alpha\tan\alpha|_0^{\pi/4} - \int_0^{\pi/4} \sec\alpha\tan^2\alpha \; d\alpha \\ &= \sqrt{2} - \int_0^{\pi/4} \sec^3\alpha - \sec\alpha \; d\alpha,\end{aligned}$$

so

$$\begin{aligned}2\int_0^{\pi/4} \sec^3\alpha \; d\alpha = \sqrt{2} + \int_0^{\pi/4} \sec\alpha \; d\alpha &= \sqrt{2} + \ln(\sec\alpha + \tan\alpha)|_0^{\pi/4} \\ &= \sqrt{2} + \ln(\sqrt{2}+1).\end{aligned}$$

Hence,

$$D = \frac{a}{\sqrt{2}} \cdot \frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right] = \boxed{\frac{a}{4}\left[2 + \sqrt{2}\ln\left(\sqrt{2}+1\right)\right]}.$$