Physics Cup 2025. Problem 2

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Task

Initially, four birds, labeled A, B, C and D are positioned at the vertices of a regular tetrahedron with side length a. At time $t = 0$ all birds begin flying. For all $t > 0$, each bird flies directly toward another bird in the following pattern: A flies toward B, B flies toward C, C flies toward D and D flies toward A. For any $t > 0$ the speeds of all the birds are equal. Calculate the total distance traveled by bird A from the start until all birds meet at a single point.

Solution

As all birds always have same velocity and we are interested only in distance traveled, we are eligible to take the velocity v of the bird constant. The easiest way to find the distance is to calculate the time before the meeting t, so the distance will be given by

$$s = \int_0^t v dt = vt$$

1. Let's note, that at the beginning

$$\frac{d(AB)}{dt} = \frac{d(BC)}{dt} = \frac{d(CD)}{dt} = \frac{d(DA)}{dt}, \qquad \frac{d(AC)}{dt} = \frac{d(BD)}{dt} \tag{1}$$

After a small period of time dt, points A, B, C, D will form 4 isosceles triangles, and as in such triangles we have two equal angles, (1) is still correct, so it will be correct for the rest of the motion, because we can continue looking at such small periods of time for the rest of the motion. That's why for our further analysis we will label $AB = BC = CD = DA = y(t)$ and $AC = BD = x(t)$. Also, because of (1) it is now obvious, that all birds will meet at the same moment of time. So all the birds travel the same distance.

2. Now let's denote the angle AB and BC form as φ, so we can derive $\cos\varphi = 1 - x^2/2y^2$ from the Cosine theorem, then derive $\sin(\varphi/2) = x/2y$ and find derivatives dx/dt, dy/dt by projecting the velocities of two birds onto the line connecting them:

$$\frac{dy}{dt} = -v(1 + \cos\varphi) = v\left(\frac{x^2}{2y^2} - 2\right) \tag{2}$$

$$\frac{dx}{dt} = -2v\sin\frac{\varphi}{2} = -v\frac{x}{y} \tag{3}$$

3. Now we have to solve this system of differential equations, which can be done the following way. Firstly, (2) has to be divided by (3) and then we need $z(x) = y/x$ substitution. We will also need to differentiate $z(x)x = y$:

$$\frac{dy}{dx} = z(x) + x\frac{dz}{dx} \tag{4}$$

Using all above we easily obtain:

$$\frac{dz}{z - \dfrac{1}{2z}} = \frac{dx}{x} \tag{5}$$

After integrating this we will get

$$\frac{1}{2}\ln|z^2 - \frac{1}{2}| = ln|x| + C \tag{6}$$

There is actually no need to use absolute value, because both expressions are positive during the motion of birds, but we will prove it later. Constant value can be expressed using boundary condition for $t = 0$, $z^2 = 1$ and $x = a$, so $C = \ln\bigl(1/a\sqrt{2}\bigr)$. Now we can get the relationship between $y(t)$ and $x(t)$:

$$2y^2(t) = x^2(t) + \frac{x^4(t)}{a^2} \tag{7}$$

Now we are able to prove, that $z^2 > 1/2$. for that we just have to check if $y = x/\sqrt{2}$ crosses (6) or not. As y and x are always positive:

$$\frac{x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sqrt{x^2 + \frac{x^4}{a^2}} \tag{8}$$

It is easy to observe, that they only cross at $x = 0$, so our suggestion was correct.

4. After substituting (6) into (3) we will have to calculate our final integral which leads us to finding t

$$-\sqrt{2}v\int_0^t dt = -\sqrt{2}vt = \int_a^0 \sqrt{1 + \frac{x^2}{a^2}}dx = I \tag{9}$$

So we have to find the value of I, which we are going to treat further as indefinite integral. We will firstly use the substitution $u = \arctan(x/a)$:

$$du = \frac{1}{a + \dfrac{x^2}{a}}, \qquad x = a\tan u, \qquad \frac{a}{\cos^2 u} = x^2 + a^2 \tag{10}$$

$$I = \int \sqrt{1 + \frac{x^2}{a^2}}dx = a\int \frac{du}{\cos^3 u} = a\int \frac{\cos u}{\cos^4 u}du \tag{11}$$

Now we will use another substitution $w = \sin u$, so $dw = \cos(u)du$

$$a\int \frac{\cos u}{\cos^4 u}du = a\int \frac{dw}{(1-w^2)^2} = \frac{a}{4}\int \left(\frac{1}{1-w} + \frac{1}{1+w}\right)^2 dw \tag{12}$$

And finally calculating the integral in terms of w:

$$\frac{a}{4}\int \left(\frac{1}{(1-w)^2} + \frac{1}{(1+w)^2} + \frac{1}{2}\left(\frac{1}{1-w} + \frac{1}{1+w}\right)\right)dw = \frac{a}{4}\left(\frac{2w}{1-w^2} + \ln\left(\frac{1+w}{1-w}\right)\right) + C \tag{13}$$

After this we can return to (7) and calculate the definite integral:

$$-\sqrt{2}vt = \frac{a}{2}\ln\left(\frac{1}{1+\sqrt{2}}\right) - \frac{a}{\sqrt{2}} \tag{14}$$

And expressing $s = vt$:

$$s = a\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\ln\left(\frac{1}{1+\sqrt{2}}\right)\right) \approx 0.81a$$