Physics Cup 2024 Problem 2

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17 November 2024

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1 Analysis of state at $t = 0$

Let us denote the magnitude of the velocities of each of the particles (We shall call birds as particles for convenience) as v. We are given that for all $t > 0$, each particle moves directly in the direction of another bird in the pattern : A follows B, B follows C, C follows D, and D follows A.

We now consider the initial state here where each side of the tetrahedron has side length a and thus each face is an equilateral triangle. We shall calculate the relative velocities of the 6 pairs of particles now.

Pairs AB, BC, CD, AD are similar to each other while pairs AC and BD are also similar to each other in respect to their relative velocities. For convenience, let subscript 1 denote the properties of the 1st set of pairs while subscript 2 denote properties of the latter set.
Relative velocities for pairs AB and similar is:

$$v_{rel1} = v + v\cos 60 = \frac{3v}{2}$$

While that for pairs AC and BD is:

$$v_{rel2} = v\cos 60 + v\cos 60 = v$$

Noting that these are not the same, we conclude that the further shapes will not necessarily be regular tetrahedrons.

2 Analysis of state at $t > 0$

Knowing that there are 2 different kinds of identical pairs, we define the sides of our tetrahedron as:

$$AC = BD = x$$

$$AB = BC = CD = AD = y$$

This can be proven using either symmetry, or induction by using the relative velocities. The proof is left to the reader.
Each face is an isosceles triangle with two of the side lengths as y and the third as x. Let us define the vertex angle to be 2θ.
Using basic geometry and trigonometry, we find the relative velocities as:

$$v_{rel1} = v + v\cos 2\theta = 2v\cos^2\theta$$

$$v_{rel2} = v\cos(90-\theta) + v\cos(90-\theta) = 2v\sin\theta$$

Writing in their differential forms, we have:

$$\frac{dx}{dt} = -v_{rel2} = -2v\sin\theta$$
$$\frac{dy}{dt} = -v_{rel1} = -2v\cos^2\theta$$

Dividing these equations we have,

$$\frac{dy}{dx} = \frac{1}{\sin\theta} - \sin\theta$$

Also,

$$\sin\theta = \frac{x}{2y}$$

Hence,

$$\frac{dy}{dx} = \frac{2y}{x} - \frac{x}{2y}$$

The solution for this differential equation is:

$$y = \sqrt{\frac{c_1x^4 + x^2}{2}}$$

Initially, $y = x = a$. Using this, we get $c_1 = \frac{1}{a^2}$.
Hence,

$$y = \sqrt{\frac{x^4 + a^2x^2}{2a^2}}$$

3 Calculating the distance traveled by A

We know from the above equations that,

$$\frac{dx}{dt} = -2v\sin\theta = -v\frac{x}{y}$$

Hence,

$$\frac{dx}{dt} = -v\sqrt{\frac{2a^2}{x^2 + a^2}}$$

So,

$$\frac{1}{\sqrt{2a^2}}\int_x^a \sqrt{x^2 + a^2}\,dx = v\int_0^t dt = vt$$

But the distance traveled by particle A is vt_{final}, where $t = t_{final}$ is the time when $x = 0$ i.e. the particles collide!

Note: When $x = 0$, y is also 0 and hence we can state all the particles collide at the same moment
We can set the limits of integration from 0 to a and set the LHS to d, where d is the distance traveled by A.
Hence,

$$\frac{1}{\sqrt{2a^2}} \int_0^a \sqrt{x^2 + a^2} dx = d$$

Carrying out this integration, we finally get,

$$d = \frac{\ln{(1 + \sqrt{2})} + \sqrt{2}}{2\sqrt{2}} a$$

In numerical values,

$$d = 0.81161a$$