Physics Cup – TalTech 2025 – Problem 2

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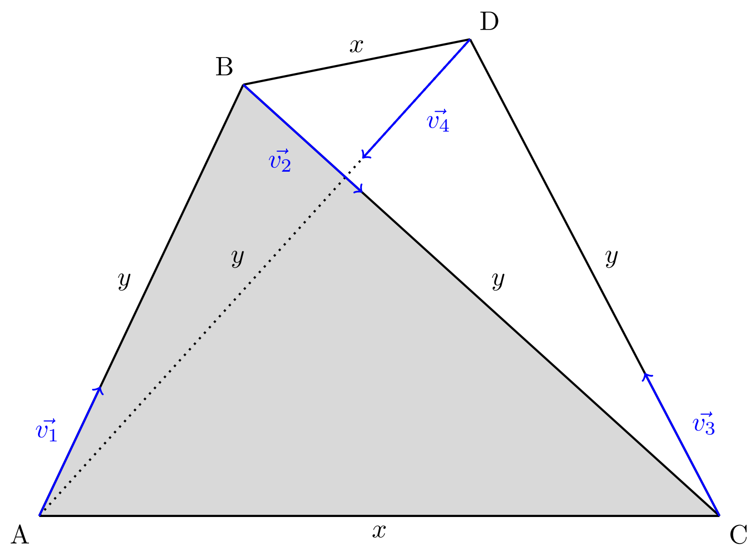

First, by symmetry, it is always true that $AB = BC = CD = DA = y$ and $AC = BD = x$. Let $\angle ABC = \angle BCD = \angle CDA = \angle DAB = \theta$, and $|\vec{v_1}| = |\vec{v_2}| = |\vec{v_3}| = |\vec{v_4}| = v$. To calculate $\frac{\mathrm{d}y}{\mathrm{d}t}$, take side AB for example. The radial speed of B with respect to A is $v + v\cos\theta$, so

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -v(1+\cos\theta) = -2v\left(1-\sin^2\frac{\theta}{2}\right) = -2v\left(1-\left(\frac{x}{2y}\right)^2\right) \tag{1}$$

For $\frac{\mathrm{d}x}{\mathrm{d}t}$, take side AC for example. Consider the projections of $\vec{v_1}$ and $\vec{v_3}$ onto AC, both of which have magnitude $v\sin\frac{\theta}{2}$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2v\sin\frac{\theta}{2} = -\frac{x}{y}v \tag{2}$$

Dividing (1) by (2),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{y}{x}\left(1-\left(\frac{x}{2y}\right)^2\right) = \frac{2y}{x} - \frac{x}{2y}$$

Let $u = \frac{y}{x}$, then

$$\begin{aligned}
\frac{\mathrm{d}(ux)}{\mathrm{d}x} &= 2u - \frac{1}{2u} \\
u + x\frac{\mathrm{d}u}{\mathrm{d}x} &= 2u - \frac{1}{2u} \\
x\frac{\mathrm{d}u}{\mathrm{d}x} &= u - \frac{1}{2u} \\
\frac{\mathrm{d}u}{u - \frac{1}{2u}} &= \frac{\mathrm{d}x}{x} \\
\frac{2u\mathrm{d}u}{2u^2 - 1} &= \frac{\mathrm{d}x}{x} \\
\int \frac{\mathrm{d}u^2}{2u^2 - 1} &= \int \frac{\mathrm{d}x}{x} \\
\frac{\ln(2u^2 - 1)}{2} &= \ln x + C \\
\sqrt{2u^2 - 1} &= Cx \\
\sqrt{2\left(\frac{y}{x}\right)^2 - 1} &= Cx \\
\sqrt{2y^2 - x^2} &= Cx^2
\end{aligned}$$

Using the initial conditions $x = y = a$, $C = \frac{1}{a}$.

$$\begin{aligned}
\sqrt{2y^2 - x^2} &= \frac{x^2}{a} \\
2y^2 - x^2 &= \frac{x^4}{a^2} \\
y &= x\sqrt{\frac{x^2 + a^2}{2a^2}}
\end{aligned}$$

To determine how long it takes for the birds to meet, we will find the time it takes for x to reach 0.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{x}{y}v \\
\sqrt{\frac{x^2+a^2}{2a^2}}\frac{\mathrm{d}x}{\mathrm{d}t} &= -v \\
\sqrt{\left(\frac{x}{a}\right)^2+1}\mathrm{d}\left(\frac{x}{a}\right) &= -\frac{\sqrt{2}}{a}v\mathrm{d}t \\
\int_1^0 \sqrt{\left(\frac{x}{a}\right)^2+1}\mathrm{d}\left(\frac{x}{a}\right) &= -\int_0^t \frac{\sqrt{2}}{a}v\mathrm{d}t \\
\left[\frac{\ln\left(\sqrt{x^2+1}+x\right)+x\sqrt{x^2+1}}{2}\right]_0^1 &= \frac{\sqrt{2}}{a}vt \\
\frac{\ln\left(\sqrt{2}+1\right)+\sqrt{2}}{2} &= \frac{\sqrt{2}}{a}vt \\
vt &= \frac{\ln\left(\sqrt{2}+1\right)+\sqrt{2}}{2\sqrt{2}}a
\end{aligned}$$

This means bird A travels $\frac{\ln\left(\sqrt{2}+1\right)+\sqrt{2}}{2\sqrt{2}}a \approx 0.812a$.