Physics Cup 2025 – Problem 2

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1 Introduction

1.1 Notation

Here are some comments about the used notation:

1. Vectors are indicated by an arrow above. They are always given in cartesian form.
2. Prime quantities are the derivatives with respect to time t. A derivative with respect to any other parameter is indicated explicitly. For column vectors, the derivative is taken on each element of the vector.
3. The time dependency of quantities is not always explicitly indicated.
4. Juxtaposition of matrices (including vectors) denotes the usual matrix multiplication.

1.2 Problem

The following quantity will be used: $\frac{a}{2\sqrt{2}} = \tilde{a}$. The trajectories are called $\overrightarrow{r_i}(t) = (x_i(t), y_i(t), z_i(t))^T$ for $i \in \{A, B, C, D\}$. We consider the following initial conditionts, which will simplify some considerations in .

$$\begin{aligned} \overrightarrow{r_A}(0) &= \tilde{a}(1,1,1)^T \\ \overrightarrow{r_B}(0) &= \tilde{a}(1,-1,-1)^T \\ \overrightarrow{r_C}(0) &= \tilde{a}(-1,1,-1)^T \\ \overrightarrow{r_D}(0) &= \tilde{a}(-1,-1,1)^T \end{aligned} \tag{1}$$

Note that the center of the tetrahedron defined by these four points is located at the origin.

2 General observations

2.1 Symmetric considerations

2.1.1 Distance between birds

The distance between a pair of birds, in one which is attracted by another, (i.e. (A, B), (B, C), (C, D) or (D, A)) is the same for all pairs at a given time. This is due to fact that one can relabel the birds in the initial conditions, but the problem will still be the same up to a rotation of the frame of reference (or equivalently, up to a relabeling of the axis).

The same argument justifies why the distance travelled by any bird until they all meet is the same.

And again the same argument, justifies why the distances between the pairs of birds A, C and (B, D) is always the same. However, note that this distance will not be the same as the distance between the 4 other pairs. This is the reason why the birds don't form a regular tetrahedron at all time, and why the answer isn't simply $3a/4$.

2.1.2 Trajectories of the birds

Consider the initial conditions given by . They are left invariant (modulo relabeling of the birds) under a rotation of 90 deg around the y-axis, which implied that for all $t > 0$

$$\begin{aligned} x_A(t) &= z_D(t) \\ x_B(t) &= z_A(t) \\ x_C(t) &= z_B(t) \\ x_D(t) &= z_C(t) \end{aligned} \tag{2}$$

Furthermore, the initial conditions are invariant under reflection over the y-axis, so for all $t > 0$

$$\begin{aligned} y_A(t) &= y_C(t) \\ y_B(t) &= y_D(t) \end{aligned} \tag{3}$$

2.2 Uniqueness of the trajectory

Because every bird shares the same speed at a given moment, the birds move as described by the following system of differential equations:

$$\begin{cases} \overrightarrow{r_A}'(t) = \dfrac{\overrightarrow{r_B} - \overrightarrow{r_A}}{\|\overrightarrow{r_B} - \overrightarrow{r_A}\|} v(t) \\ \overrightarrow{r_B}'(t) = \dfrac{\overrightarrow{r_C} - \overrightarrow{r_B}}{\|\overrightarrow{r_C} - \overrightarrow{r_B}\|} v(t) \\ \overrightarrow{r_C}'(t) = \dfrac{\overrightarrow{r_D} - \overrightarrow{r_C}}{\|\overrightarrow{r_D} - \overrightarrow{r_C}\|} v(t) \\ \overrightarrow{r_D}'(t) = \dfrac{\overrightarrow{r_A} - \overrightarrow{r_D}}{\|\overrightarrow{r_A} - \overrightarrow{r_D}\|} v(t) \end{cases} \tag{4}$$

The trajectory of the birds will be independent of their velocity. The intuitive reason would be that, since the birds are always moving with the same speed $v(t)$, the distance they have traveled will always be the same at any time t. So we can just take $v(t) = \|\overrightarrow{r_B} - \overrightarrow{r_A}\|$, and work with the following system of differential equations:

$$\begin{cases} \overrightarrow{r_A}' = \overrightarrow{r_B} - \overrightarrow{r_A} \\ \overrightarrow{r_B}' = \overrightarrow{r_C} - \overrightarrow{r_B} \\ \overrightarrow{r_C}' = \overrightarrow{r_D} - \overrightarrow{r_C} \\ \overrightarrow{r_D}' = \overrightarrow{r_A} - \overrightarrow{r_D} \end{cases} \tag{5}$$

2.3 Meeting point

Adding up all the equations in the system, yields:

$$\overrightarrow{r_A}' + \overrightarrow{r_B}' + \overrightarrow{r_C}' + \overrightarrow{r_D}' = (\overrightarrow{r_A} + \overrightarrow{r_B} + \overrightarrow{r_C} + \overrightarrow{r_D})' = 0 \tag{6}$$

Imagining that all birds have mass equal to 1 unit, this shows that the momentum of the birds is conserved throughout their movement. In other words, their center of mass doesn't change. Therefore, if the birds meet, they must meet at their initial center of mass, which is the geometric center of the initial tetrahedron, i.e. the origin.

3 Solution

Using the observations made before, the system can be simplified down to the following 6 equations:

$$\begin{cases} z'_D = x'_A = z_A - x_A & (i) \\ y'_C = y'_A = y_B - y_A & (ii) \\ x'_B = z'_A = z_B - z_A & (iii) \\ y'_D = y'_B = y_A - y_B & (iv) \\ x'_C = z'_B = z_C - z_B & (v) \\ x'_D = z'_C = x_A - z_C & (vi) \end{cases} \tag{7}$$

Equations (ii) and (iv) are independent form the other 4. Adding them together yields

$$(y_A + y_B)' = 0$$

so $y_A + y_B$ is a constant for all $t > 0$, i.e.

$$y_A(t) + y_B(t) = y_A(0) + y_B(0) = 0 \implies y_B = -y_A$$

Plugging this result back into (ii) gives

$$y'_A = -2y_A.$$

whose general solution is

$$y_A(t) = Ce^{-2t}$$

where C is a real constant. Initial conditions imply that $C = \tilde{a}$, and thus

$$y_A(t) = \tilde{a}e^{-2t}, \quad y_B(t) = -\tilde{a}e^{-2t}.$$

Now let's turn our attention to the remaining 4 equations. Adding them all up yields

$$(x_A + z_A + z_B + z_C)' = 0$$

so

$$x_A(t) + z_A(t) + z_B(t) + z_C(t) = 0 \implies z_A + z_C = -(x_A + z_B)$$

Plugging it back into $(i) + (v)$ gives

$$(x_A + z_B)' = -2(x_A + z_B)$$

so, because of the initial conditions:

$$x_A(t) + z_B(t) = 0 \implies x_A = -z_B, \quad z_A(t) + z_C(t) = 0 \implies z_A = -z_C.$$

reduces to

$$\begin{cases} x_A' = z_A - x_A \\ z_A = -z_A - x_A \end{cases}$$

The solution of which is given by:

$$\begin{cases} x_A(t) = Ae^{-t}\cos t + Be^{-t}\sin t \\ z_A(t) = -Ae^{-t}\sin t + Be^{-t}\cos t \end{cases}$$

Initial conditions give $A = B = \tilde{a}$. So the trajectory of the bird A is

$$\overrightarrow{r_A}(t) = \tilde{a}\begin{pmatrix} e^{-2t} \\ e^{-t}(\cos t + \sin t) \\ e^{-t}(\cos t - \sin t) \end{pmatrix} \tag{8}$$

$$\overrightarrow{r_A}'(t) = \tilde{a}\begin{pmatrix} -2e^{-2t} \\ -2e^{-t}\sin t \\ -2e^{-t}\cos t \end{pmatrix} \tag{9}$$

The total distance traveled by the bird A is:

$$L_A = \int_0^{+\infty} \|\overrightarrow{r_A}'(t)\| \mathrm{d}t = 2\tilde{a}\int_0^{+\infty} e^{-t}\sqrt{e^{-2t} + 1}\mathrm{d}t \tag{10}$$

The value of this integral can be computed with the substitution $u = e^{-t}$ for example, yielding the final result:

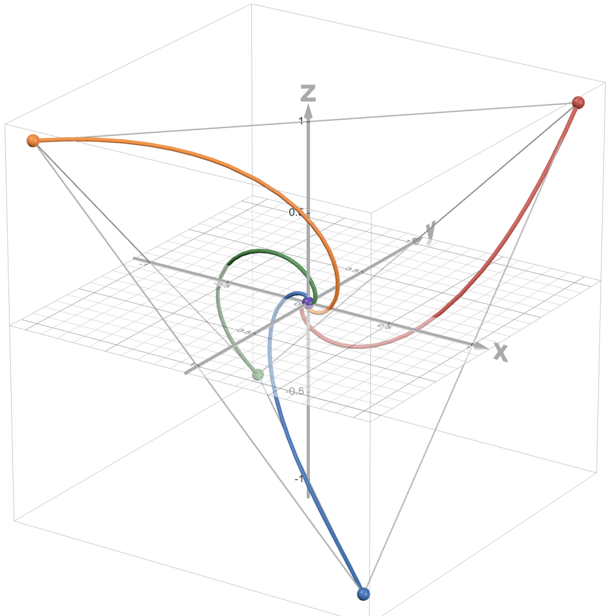
$$\boxed{\frac{a}{2}\left(1 + \frac{\operatorname{arcsinh}(1)}{\sqrt{2}}\right)} \tag{11}$$

or equivalently:

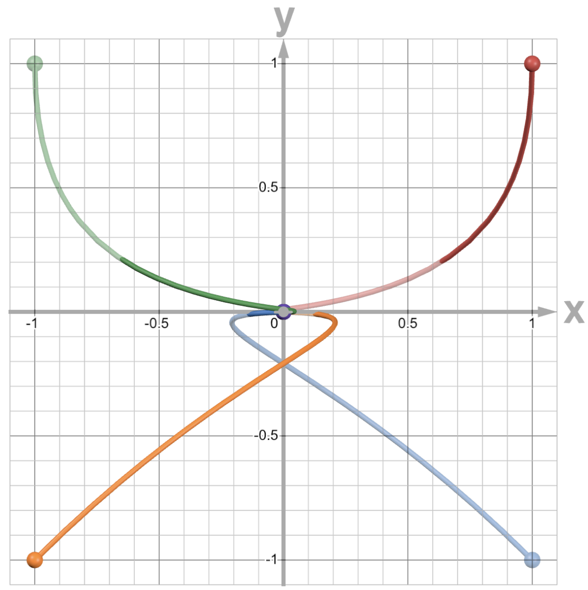
$$\boxed{\frac{a}{2}\left(1 + \frac{\ln\left(1 + \sqrt{2}\right)}{\sqrt{2}}\right)} \tag{12}$$

4 Figures

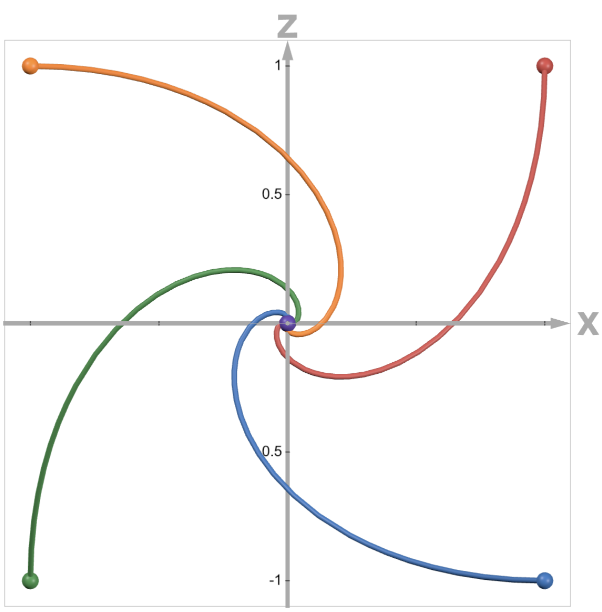
A small simulation in Desmos can be found https://www.desmos.com/3d/a2etyfh2dlhere. We include the following figures for illustrative purposes.



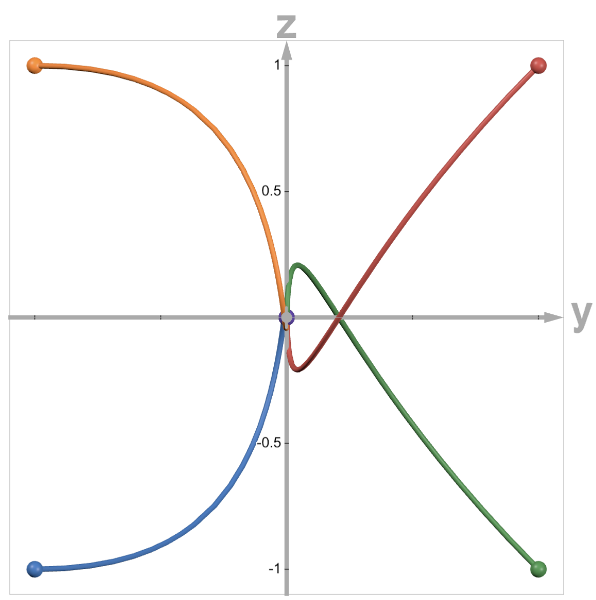
(a) 3D view.



(b) Projection on the x-y plane.



(c) Projection on the x-z plane.



(d) Projection on the y-z plane.

Figure 1: Visualization of the trajectories of the 4 birds: A – red, B - blue, C – green, D – orange; for a side length of $a = 2\sqrt{2}$. The initial conditions are given by .