Physics Cup – TalTech 2025

Emerson Franzua Aldana Gavarrete

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1 Problem 3: Oscillating Piston

A monoatomic gas fills a cylinder of height H under a piston. The piston oscillates periodically up and down with amplitude a: during the first half-period, it moves upward with a constant speed u, and during the second half-period, it moves downward with the same constant speed u. Initially, the root-mean-square (RMS) speed of the molecules is v. How much time t will it take for the RMS speed to double? Use the following model assumptions: the walls and the piston are perfect heat insulators and have zero heat capacity; the surface of the piston is perfectly flat; the mean free path of the molecules λ satisfies the following conditions: $H \gg \lambda \gg a$ and $\lambda \gg \frac{H^2}{vt}$. Additionally, $v \gg u$.

2 Initial Considerations

Let A represent the piston face, B the opposite face, S the cylinder's cross-sectional area, T_0 the initial temperature, n the particle density, m the particle mass, and v_{rms} the root-mean-square (RMS) speed of the gas particles, with $v_{rms}(0) = v$ and $v_{rms}(t) = 2v$. Additionally, based on the model assumptions, the following simplifications are made:

- *Collisions with the piston are wall collisions and without scattering.* The piston is modeled as a perfect heat insulator with a perfectly flat surface and zero heat capacity. Thus, it does not absorb energy from the particles. In addition, the piston's speed remains unchanged after collisions. Therefore, in the rest frame of the piston, it behaves as a perfect elastic wall.
- *Collisions between particles are negligible in the oscillation region.* This assumption is valid because $\lambda \gg a$. As a result, particle-particle collisions are unlikely to occur in this region.
- *The gas is ideal and remains in thermal equilibrium.* Since $H \gg \lambda$ the gas is long enough to have a continuous temperature gradient. By this fact, and the additional condition, $\lambda \gg \frac{H^2}{vt}$ it can be shown that the temperature variation is negligible (see appendix for justification).

3 Heat Generation

Fact 1. *A particle with horizontal speed v_x[1] gains or loses energy $\Delta K = 2mu(u - v_x)$, where v_x and u points in the same direction.*

Proof. In the frame where the piston is stationary, the particle's speed changes to $v_x - u$. After the collision, it reflects to $u - v_x$. Returning to the cylinder frame, the speed becomes $2u - v_x$. Hence, the change in kinetic energy is:

$$\Delta K = \frac{1}{2}m\left[(2u - v_x)^2 - v_x^2\right] = 2mu(u - v_x).$$

□

To compute the energy gain over a period of oscillation $2T = 2a/u$, note that the horizontal velocity component (perpendicular to the piston), v_x, follows a one-dimensional Maxwell distribution $f(v_x)$. Consider a particle with speed v_x. It will collide with the piston during its forward motion[2] over a time interval t_1, and during its return motion over a time interval $t_2 = T - t_1$. The time t_1 is determined by the condition:

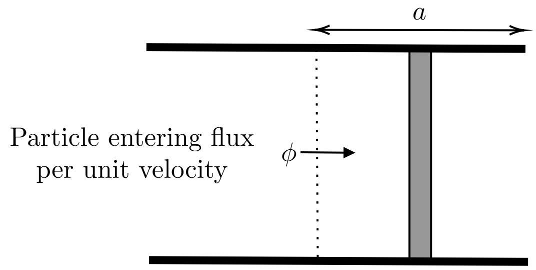
$$v_x\left(T - t_1\right) = uT \Rightarrow t_1 = T\left(1 - \frac{u}{v_x}\right),$$

and, therefore, the time for the return motion is:

$$t_2 = T\left(1 + \frac{u}{v_x}\right).$$

[1]The subscript indicates the component along u, and avoids confusion with v.

[2]Forward motion is characterized by a positive speed u, and return motion by a negative speed $-u$.



Boundary

The particle density at the boundary (see figure) obeys a speed distribution $\frac{dn}{dv_x} = nf(v_x)$. Hence, the particle flux distribution is given by $\phi(v_x) = nSv_x f(v_x)$ [3]. If the energy gain per collision is ΔK_1 and ΔK_2 for the forward and return motion respectively, the average generated heat current ($\dot{Q}$) is given by:

$$d\dot{Q} = nSv_x f(v_x)\left(\frac{E_1 t_1 + E_2 t_2}{2T}\right) dv_x.$$

Using Fact 1 and replacing t_1 and t_2,

$$d\dot{Q} = (mu)nSv_x f(v_x)\left((u - v_x)\left(1 - \frac{u}{v_x}\right) + (v_x + u)\left(1 + \frac{u}{v_x}\right)\right) dv_x \Rightarrow d\dot{Q} = 4nmSu^2 v_x f(v_x).$$

Therefore,

$$\dot{Q} = 4nmSu^2 \int_0^\infty v_x \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{v_x^2}{2\sigma^2}\right)} dv_x; \quad \sigma = \sqrt{\frac{kT}{m}}.$$

$$\dot{Q} = \frac{4}{\sqrt{2\pi}} nSu^2 \sqrt{km}\sqrt{T}.$$

4 Differential Equation

Since $v = \sqrt{\frac{3kT_0}{m}}$, increasing v to $2v$ require raising the temperature from T_0 to $4T_0$. The heat capacity is $C_v = \frac{3}{2}knSH$, giving:

$$\frac{3}{2}knSH\frac{dT}{dt} = \frac{4}{\sqrt{2\pi}} nSu^2\sqrt{km}\sqrt{T}.$$

Replacing $\sqrt{m/k} = \sqrt{3T_0}/v$, we obtain:

$$\int_{T_0}^{4T_0} \frac{dT}{\sqrt{T}} = \frac{8}{\sqrt{6\pi}}\frac{u^2}{vH}\sqrt{T_0}\int_0^t dt \Rightarrow 2\left(\sqrt{4T_0} - \sqrt{T_0}\right) = \frac{8}{\sqrt{6\pi}}\frac{u^2}{vH}\sqrt{T_0}t.$$

Thus,

$$\boxed{t = \frac{\sqrt{6\pi}}{4}\frac{vH}{u^2}.}$$

5 Appendix: Justification of Thermal Equilibrium

To double the RMS speed, the total heat Q should quadruple the original temperature T_0, hence we can estimate $Q \approx \frac{9kNT_0}{2}$. The average heat current generated by the piston is $\langle I_0 \rangle = \frac{Q}{t}$, the average time between collisions is $\langle \tau \rangle = \frac{\lambda}{v_{\text{rms}}}$, and the average horizontal distance traveled between collisions is $\langle \Delta x \rangle = \frac{\lambda}{\sqrt{3}}$. Hence, the average current flux inside the gas is [4]:

$$\langle I_{\text{in}} \rangle = \frac{1}{2\langle \tau \rangle} kN_\lambda \left\langle \frac{dT}{dx} \right\rangle \langle x \rangle = \frac{1}{6} kN\frac{\lambda}{H}\left\langle \frac{dT}{dx} \right\rangle v_{\text{rms}} < \langle I_0 \rangle = \frac{9}{2t}kNT_0.$$

Since we can compute $\Delta\langle T \rangle = \langle T_A - T_B \rangle = \left\langle \frac{dT}{dx} \right\rangle H$, it follows that:

$$\frac{\langle \Delta T \rangle}{T_0} < \frac{27}{\lambda}\frac{H^2}{v_{\text{rms}}t} < \frac{27}{\lambda}\frac{H^2}{vt} \ll 1.$$

Thus, we have established that the gas can be approximately treated as if it were in equilibrium.

[3] i) $\phi(v_x)$ represents the particles' flux crossing the boundary with speeds between v_x and $v_x + dv_x$. ii) To see why the factor v_x, consider the particles crossing the boundary during a time dt. They come from a region $v_x dt$ behind the boundary. Dividing by dt, you obtain the expected result.

[4] Here, $\langle\ \rangle$ represent average over time and space, and N_λ is the number of molecules in a volume S$\langle x \rangle$. All molecules in that volume are going to collide on an average time $\langle \tau \rangle$.