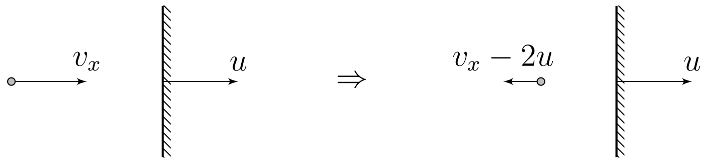
To solve the problem we will discuss two processes in the cylinder: the transfer of energy from the piston to the layer of gas close to it (thickness of the layer $\approx \lambda$); the transfer of heat from this layer to the other gas. This devision is possible because of $\lambda \ll H$.

Thin layer heating

Let's focus on the molecule bouncing off the surface with area dS on the wall which moves with the velocity u:



The probability $\mathrm{d}P$ of the event when the molecule with velocity in the range $[v_x, v_x + \mathrm{d}v_x]$ bouncing during $\mathrm{d}t$ is

$$\mathrm{d}P = nf(v_x)\mathrm{d}v_x \cdot v_x \mathrm{d}t\mathrm{d}S \cdot \theta\left[v_x > u\right],$$

where $\theta(x)$ is 1 when condition x is true and 0 when x is false, $f(v_x)$ is the PDF of the v_x, n is the concentration of the molecules.

This relation is possible only because of $a \ll \lambda$ and $u \ll v$, i.e. molecules don't have to catch up the wall.

Then the change of the momentum projection $\mathrm{d}p$ of the molecules with velocity in the range $[v_x, v_x + \mathrm{d}v_x]$ is

$$\mathrm{d}p = m(-v_x + 2u - v_x)\mathrm{d}P = -2m(v_x - u)\mathrm{d}P,$$

where m is the mass of the molecule. The change of the energy of the molecules with velocity in the range $[v_x, v_x + \mathrm{d}v_x]$ is

$$\mathrm{d}E = m\frac{(v_x - 2u)^2 - v_x^2}{2}\mathrm{d}P = 2mu\left(u - v_x\right)\mathrm{d}P.$$

Work of the piston

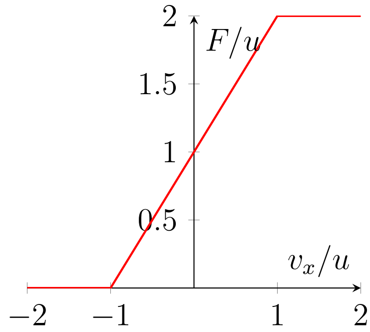
The pressure $\mathcal{P}(u)$ which acts on the wall is:

$$\mathcal{P}(u) = -\int \frac{\mathrm{d}p}{\mathrm{d}t\mathrm{d}S} = mn\int\limits_{-\infty}^{+\infty} f(v_x)v_x\mathrm{d}v_x \cdot 2(v_x - u)\theta[v_x > u]$$

So if S is the area of the piston we can find a work A of the piston on the thin boundary layer during one period :

$$\frac{A}{aS} = -\mathcal{P}(u) + \mathcal{P}(-u) = 2mn\int\limits_{-\infty}^{+\infty} f(v_x)v_x\mathrm{d}v_x \underbrace{\left(-(v_x - u)\theta[v_x > u] + (v_x + u)\theta[v_x > -u]\right)}_{F(v_x)}$$

The function $F(v_x)$ is drawn below:



Thus $F(v_x) = 2u + F_1(v_x)$, where $F_1(v_x) \neq 0$ only in range $(-u, u)$ and

$$\frac{A}{2mnaS} \simeq 2u \int\limits_0^{+\infty} f(v_x) v_x \mathrm{d}v_x$$

because $u \ll v$. For the maxwellian distribution $f(v_x)$ with the temperature T we have

$$\frac{A}{2mnaS} = u\sqrt{\frac{m}{2\pi kT}} \int\limits_0^{+\infty} \frac{2kT}{m} e^{-\frac{mv^2}{2kT}} \mathrm{d}\left(\frac{mv^2}{2kT}\right) = u\sqrt{\frac{2kT}{\pi m}}$$

Finally, the change of the internal energy of the boundary layer is $\Delta U = A$. For the total volume of gas it could be described as the power W which supplied on the surface of the piston

$$W = \frac{Au}{4a} = \frac{mnSu^2}{2}\sqrt{\frac{2kT}{\pi m}}.$$

The same result could be obtained with the integration of $\mathrm{d}E$ over velocities of the molecules.

Heat conduction in the gas

For the heat conduction problem we know the boundary conditions: zero power in the $x = 0$ ($\frac{\partial T}{\partial x} = 0$) and given power in the $x = H$ ($W = \varkappa S \frac{\partial T}{\partial x}$). It allows us to estimate the difference in the temperature on the ends of gas:

$$\Delta T \approx \frac{WH}{\varkappa S}$$

and the equation: For the gas we have $\varkappa \approx \rho\lambda c\sqrt{kT/m}$, where $\rho = mn$, $c \approx k/m$, so $\varkappa \approx kn\lambda\sqrt{kT/m}$. The estimation for ΔT:

$$\Delta T \approx \frac{mnu^2 H\sqrt{kT/m}}{\lambda kn\sqrt{kT/m}} = \frac{mu^2 H}{k} \approx \frac{u^2 H}{\lambda v^2} T \ll T.$$

That is the temeperature of the gas is constant over the coordinate.

Gas heating

The power W heats the gas:

$$\frac{3}{2}\nu R\dot{T} = W = \frac{mnSu^2}{2}\sqrt{\frac{2kT}{\pi m}},$$

where $\nu = nSH/N_A$, so

$$3Hk\dot{T} = mu^2\sqrt{\frac{2kT}{\pi m}} \quad \Rightarrow \quad \frac{\mathrm{d}T}{\sqrt{T}} = \frac{mu^2}{3Hk}\sqrt{\frac{2k}{\pi m}}\mathrm{d}t.$$

As $v \propto \sqrt{T}$, $T_{\text{final}} = 4T_0$ and

$$\sqrt{T_0} - \frac{1}{2}\sqrt{T_0} = \frac{mu^2}{3Hk}\sqrt{\frac{2k}{\pi m}}t, \quad \Rightarrow \quad t = \frac{3H}{2u^2}\sqrt{\frac{\pi k T_0}{2m}}$$

With $v = \sqrt{3kT/m}$ we have

$$t = \frac{vH}{2u^2}\sqrt{\frac{3\pi}{2}}$$