Physics Cup Problem 3

Oscillating piston

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1 Qualitative description

1.1 Change of internal energy

The piston's oscillation causes the molecules of the gas that collide with it to change their kinetic energy. The process is divided into two half-periods.

The relation $\lambda >> \frac{H^2}{vt}$ can be transformed into:

$$\frac{d}{H} >> \frac{H}{\lambda} \tag{1}$$

Where $d = vt$ is the total distance done by a molecule, assuming average kinetic energy. This says that the order of the number of vertical travels (left) is much higher than the order of the number of collisions per travel (right).

From this we understand that the molecules mix up quickly within the cylinder, so we can add the obtained kinetic energy of one molecule, ΔE, to the internal energy of the gas $U(T)$, thus changing the temperature of the whole gas, along with the velocities of the molecules.

$$dU = N_{collided}\Delta E \tag{2}$$

1.2 Speed of the molecules

In this solution v will signify the speed of one molecule on the vertical axis of the cylinder, pointing upwards.

To analyze the molecules hitting the wall, we will use the Maxwell-Boltzmann distribution, i.e. the probability of one molecule having a velocity between v and $v + dv$ on one axis:

$$f(v)dv = \sqrt{\frac{m}{2\pi k_B T}} e^{\frac{-mv^2}{2k_B T}} dv \tag{3}$$

Later, using this, we will integrate for all possible velocities.

2 Quantitative solution

The following constants will be defined:

- A - the cross-section of the cylinder and piston;
- N - the total number of molecules in the gas;
- m - the mass of one molecule;
- T_n - the temperature of the gas at the start of one oscillation.

2.1 First half-period

This is the part where the piston moves upwards with speed u.

2.1.1 Change of kinetic energy

The speed of the molecules that hit the piston will become $v - 2u$, for every molecule with a velocity $v > u$.

$$\Delta E = \frac{m}{2}((v - 2u)^2 - v^2) = 2mu(u - v) \tag{4}$$

2.1.2 Number of molecules that collide

The molecules traveling towards the piston in time dt will sweep out a volume $A(v-u)dt$ - in the piston's frame of reference. The number of molecules that collide with the piston in time dt, having velocity between v and $v+dv$ will therefore be:

$$dN_{collided(v)} = A(v-u)dt \cdot \frac{N}{V} \cdot f(v)dv \tag{5}$$

2.1.3 Differential equation for T

From Equations 4 and 5, and the equation for internal energy $dU = \frac{3}{2}Nk_B dT$, we obtain:

$$\frac{3}{2}Nk_B dT = 2mu(u-v) \cdot A(v-u)dt\frac{N}{V}f(v)dv \tag{6}$$

However, we have to account for every speed from u to ∞, and $V = Ah$:

$$3Nk_B dT = -4mu\frac{dt}{h}\int_u^\infty (v-u)^2 f(v)dv \tag{7}$$

The same thing is done for the other half-period.

2.2 Second half period

This is the part where the piston moves downwards with speed u.

The speed of the molecules that hit the piston will become $v+2u$, for every molecule with a velocity $v > -u$.

$$\Delta E = \frac{m}{2}((v+2u)^2 - v^2) = 2mu(v+u) \tag{8}$$

Next we will find the number of molecules that collide:

$$dN_{collided(v)} = A(v+u)dt \cdot \frac{N}{V} \cdot f(v)dv \tag{9}$$

From Equations 8 and 9, we will obtain the differential equation for T:

$$3Nk_B dT = 4mu\frac{dt}{h}\int_{-u}^\infty (v+u)^2 f(v)dv \tag{10}$$

2.3 Solution to the integrals

Both the differential equations for T contain similar integrals that need to be expressed as a function of T.

$$\begin{aligned} \int_u^\infty (v-u)^2 f(v)dv &= \sqrt{\frac{m}{2\pi k_B T}}\int_u^\infty (v-u)^2 e^{\frac{-mv^2}{2k_BT}}dv \\ \int_{-u}^\infty (v+u)^2 f(v)dv &= \sqrt{\frac{m}{2\pi k_B T}}\int_{-u}^\infty (v+u)^2 e^{\frac{-mv^2}{2k_BT}}dv \end{aligned} \tag{11}$$

2.3.1 Approximating the integrals

Expanding the expressions $(v \pm u)^2$ we obtain $v^2 + u^2 \pm 2uv$, getting the sum of three integrals of the form:

$$\int x^2 e^{-ax^2}dx; \quad \int xe^{-ax^2}dx; \quad \int e^{-ax^2}dx$$

To solve these integrals, we will change the lower bounds of the integrals, from u and $-u$, to 0. This way we are ignoring molecules having a velocity in the intervals $(0, u)$ and $(-u, 0)$. We can do that because of the approximation $v >> u$.

2.3.2 Solving through Feynmen's trick

The solution of these integrals is therefore:

$$\sqrt{\frac{m}{2\pi k_B T}}\int_{-u}^{\infty}(v\pm u)^2 e^{\frac{-mv^2}{2k_B T}}dv = \frac{k_B T}{2m} \pm 2u\sqrt{\frac{k_B T}{2\pi m}} + \frac{u^2}{2} \approx \alpha T + \beta\sqrt{T} \tag{12}$$

Because $v >> u$ and the order of the first two terms is v^2 and v, we can ignore the third term $u^2/2$. $\alpha = \frac{k_B}{2m}$ and $\beta = 2u\sqrt{\frac{k_B}{2\pi m}}$.

2.4 Final differential equations

Plugging equations 12 into equations 7 and 10, we obtain the integrals:

$$\begin{aligned} \int_{T_n}^{T'} \frac{3k_B dT}{\alpha T - \beta\sqrt{T}} &= -4mu\int_0^{\frac{a}{u}} \frac{dt}{H+ut} \\ \int_{T'}^{T_{n+1}} \frac{3k_B dT}{\alpha T + \beta\sqrt{T}} &= 4mu\int_0^{\frac{a}{u}} \frac{dt}{H+ut} \end{aligned} \tag{13}$$

From which we can solve for $T_{n+1}(T_n)$, where $c = (\frac{H+a}{H})^{-1/3}$:

$$\begin{aligned} &\sqrt{T'} = \frac{\alpha c\sqrt{T_n} + \beta(1-c)}{\alpha} \\ &\alpha c\sqrt{T_{n+1}} + \beta c = \alpha\sqrt{T'} + \beta \\ &\alpha c\sqrt{T_{n+1}} = \alpha c\sqrt{T_n} + 2\beta(1-c) \\ &\frac{1}{c} = (\frac{H+a}{H})^{\frac{1}{3}} \approx 1 + \frac{a}{3H} \\ &\sqrt{T_{n+1}} = \sqrt{T_n} + \frac{2\beta a}{3\alpha H} \end{aligned} \tag{14}$$

Therefore, after n oscillations, the square root of the temperature will be:

$$\sqrt{T} = \sqrt{T_0} + \frac{2\beta a n}{3\alpha H} \tag{15}$$

Since v is proportional to $\sqrt{T}$, it will double when $\sqrt{T} = 2\sqrt{T_0}$, and $\sqrt{T_0} = v\frac{m}{3k_B}$

$$\begin{aligned} &v\frac{m}{3k_B} = \frac{2\beta a n}{3\alpha H} \\ &n = \frac{\sqrt{6\pi} v H}{8au} \\ &t = \frac{2a}{u}n = \frac{\sqrt{6\pi} v H}{4u^2} \\ &t \approx 1.085\frac{vH}{u^2} \end{aligned} \tag{16}$$