Physics Cup 2025 – Problem 3

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1 Introduction

Notation and considerations

- v is the RMS speed, as a function of time. The time dependency is often omitted for the sake of clear notation. It will be indicated explicitly when the distinction between v and $v(t)$ is needed.
- The x direction is taken along the axial axis of the cylinder, perpendicular to the surface of the piston, with the positive direction upwards.
- The amplitude a is taken to be half the peak to peak value in this solution. Taking a to be the peak to peak amplitude will change some quantities, but not the final solution. The piston oscillates with a period τ given by $\tau = 2 \cdot 2a/u = 4a/u$.
- N is the total number of gas particles, m is the mass of a single particle and k_B is Boltzmann constant.

Simplifications The given assumptions on the model are used to make the following simplifications:

- *the walls and the piston are perfect heat insulators and have zero heat capacity, the surface of the piston is perfectly flat*: The collisions between a particle and a wall of the cylinder or the piston are perfectly elastic. The incident angle is equal to the reflected angle. Conservation of momentum applies.
- $H \gg \lambda$: A particle bouncing off the piston will collide with other particles and not keep bouncing off the two ends of the cylinder and keep increasing its momentum (in the x direction). This ensures that the momentum acquired by the particles bouncing off the piston is redistributed in all the directions, and that the velocity of the particles are isotropic.
- $H \gg a$: The oscillation of the piston is negligible compared to the height of the cylinder. This means that at all times, the concentration of particles in the cylinder is approximately equal.
- $v \gg u$: The concentration of particles can be considered to be uniform at all times.
- $\lambda \gg \frac{H^2}{vt}$: Together with $H \gg \lambda$, this inequality implies that $vt \gg H$. Because the RMS speed will keep increasing (see later discussion), this ensures that the density of particles is uniform at all times, and that the distribution of the speeds of the particles can be considered uniform throughout the cylinder (i.e. it does not depend on position). (In other terms, one could say that the temperature of the gas is uniform at all times.)

2 Solution

At an instant t, the proportion of particles $\mathrm{d}N/N$ that will have a speed between v_x and $v_x + \mathrm{d}v_x$ in the x direction is given by the Maxwell–Boltzmann distribution $f(v_x)$:

$$f(v_x)\,\mathrm{d}v_x = \sqrt{\frac{3}{2\pi v^2}} \exp\left(-\frac{3v_x^2}{2v^2}\right) \mathrm{d}v_x$$

A particle bouncing off the piston whilst it is moving downwards or upwards respectively will have a new velocity $|v_x \pm 2u|$. So after a period, the RMS speed will have changed by

$$\begin{aligned} v^2(t+\tau) &= v^2(t) + \frac{\tau}{2H}\int_u^\infty (v_x - u)(v_x - 2u)^2 f(v_x)\,\mathrm{d}v_x + \frac{\tau}{2H}\int_{-u}^\infty (v_x + 2u)(v_x + 2u)^2 f(v_x)\,\mathrm{d}v_x \\ &= v^2(t) - \frac{\tau}{2H}\int_0^\infty 4uv_x^2 f(v_x + u)\,\mathrm{d}v_x + \frac{\tau}{2H}\int_0^\infty 4uv_x^2 f(v_x - u)\,\mathrm{d}v_x \\ &= v^2(t) + \frac{4\tau u}{2H}\int_0^\infty v_x^2\left(f(v_x - u) - f(v_x + u)\right)\mathrm{d}v_x \\ &\approx v^2(t) + \sqrt{\frac{3}{2\pi}}\frac{4\tau u^2}{Hv}\left(\frac{2}{3}v^2 + \frac{1}{2}u^2\right) \approx v^2(t) + \sqrt{\frac{3}{2\pi}}\frac{4\tau u^2}{H}\frac{2}{3}v \end{aligned}$$

Now, because $t \ll \tau$, one can say that:

$$v^2(t+\tau) - v^2(t) \approx \tau\frac{\mathrm{d}v^2}{\mathrm{d}t} = 2\tau v\frac{\mathrm{d}v}{\mathrm{d}t} = \sqrt{\frac{3}{2\pi}}\frac{4\tau u^2}{H}\frac{2}{3}v$$

which yields

$$v(t) = v + \sqrt{\frac{2}{3\pi}}\frac{2u^2}{H}t$$

and the final answer:

$$\boxed{t = \frac{1}{2}\sqrt{\frac{3\pi}{2}}\frac{Hv}{u^2}}$$

3 Comments on the answer

$t \gg \tau$ Using the given relations, we obtain:

$$t \sim \frac{Hv}{u^2} \gg \frac{H}{u} \gg \frac{a}{u} \sim \tau$$

Dependency on a The time it takes to double the RMS speed does not depend on the amplitude of the oscillations of the piston. This makes sense because we are using the simplification that we are always at the end of a cycle, so all that matters is the average rate at which a cycle adds energy to the system. Comparing a system with amplitude a_1 and another with amplitude $a_2 = Ca_1$, both having the same piston speed u, the first system will have done C oscillations by the time the second system does one. The total volume of gas displaced throughout this period will be equal in both cases, because the speeds of the pistons are equal. Therefore, the number of collisions in both cases will be equal. We have previously discussed that over one cycle, v can be considered constant, and since over the same time as the second system goes through one oscillation, the first goes through C, we can also approximate that the first system's v is constant throughout the C oscillations (assuming that the energy gained in later cycles is not much bigger than the first one). Both systems see the same number of collisions at the same velocity distribution, and therefore gain energy at the same rate: the value of a does not change the answer.