Physics Cup 2025 Problem 3

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January 2025

Summary

Due to the high rate of particle-particle collisions we simplify the dynamics of the particle-piston collisions such that we let all particles travel with the rms velocity v before collision. The energetics of the particle collisions with the piston and the probabilities for each kind of collision (gaining energy or losing energy) are considered. We then determine the expected energy change ΔE from each collision and determine the rate of collisions of particles with the piston wall. We then determine the total energy change of the gas in one oscillation and thereby determine the number of oscillations (and hence the time t taken) needed for the rms speed to double. In the Appendix we verify that the simplification of the dynamics is valid by solving for t in the general case where the assumption $v \gg u$ is not necessarily true.

Assumptions

1. $H \gg \lambda$ tells us that the particles collide so often that the energy gains/losses of the particles after colliding with the piston are distributed evenly throughout the gas i.e. the particles still follow a Boltzmann distribution, just that the rms speed v slowly increases with time. It also tells us that the motion of the particles inside the cylinder is so chaotic that the direction of the velocities of particles that hit the piston is effectively random.

2. Since $H \gg \lambda \gg \frac{H^2}{vt}$, we have $t \gg \frac{H}{v}$, which tells us that the timescale of the aforementioned energy redistribution is much faster than the timescale of the rms speed increasing.

3. $v \gg u$ allows us to ignore the particles that move with a z-velocity lower than u as the number of such particles will be very small. In the Appendix, we will verify that the solution for the general case (where v is not necessarily much larger than u) reduces properly in the requested limit.

4. Since the walls and the piston are perfect insulators and have zero heat capacity, we can conclude that the only changes to the rms speed are due to the dynamics of the collisions of the particles with the piston.

5. It is assumed that all collisions with the piston are perfectly elastic, so that we do not have to account for changes in the rms speed due to energy losses in inelastic collisions (which would be rather complicated).

6. The assumption $\lambda \gg a$ allows us to ignore collisions between atoms within the oscillation area of the piston.

7. It is assumed that the phrase "oscillates periodically up and down with amplitude a", means that in each half-period, the piston moves up/down by $2a$ before changing direction.

Dynamics of the Particle Collisions

Since many collisions with the piston and other particles will happen, we can simply treat the dynamics of the collision as if all the particles are moving with the rms speed v. The velocity direction of all the particles is fully random, so we can find the average upward component of velocity v_z by assuming all the velocity components are equal on average. Since there are 3 components of the velocity, v_x, v_y and v_z, and $v_x^2 + v_y^2 + v_z^2 = v^2$, we have $v_z = \frac{v}{\sqrt{3}}$.

Now let us consider the dynamics of the collisions. We have two cases, the piston moving up or the piston moving down:

1. When the piston moves up with speed u, particles approaching it with speed v_z will bounce off with a speed lower than v_z. To see this, we go into the piston's frame, where it acts as a wall since its speed is constant. In this frame, the particles have speed $v_z - u$ and bounce off with the same speed. Therefore, in the lab frame the final particle speed is $v_z - 2u$.

2. When the piston moves down with speed u the logic is similar. In the piston's frame, the particles approach and bounce off with speed $v_z + u$, so in the lab frame the final speed is $v_z + 2u$.

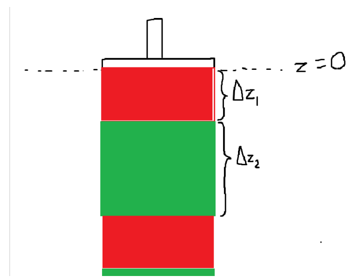
Let each particle have mass m. The energy changes for each collision (Type 1 or 2 as described above) are

$$\Delta E_1 = \frac{1}{2} m \left(v_z - 2u\right)^2 - \frac{1}{2} m v_z^2 = -\frac{1}{2} m \left(4uv_z - 4u^2\right) = -2mu(v_z - u)$$

$$\Delta E_2 = \frac{1}{2} m \left(v_z + 2u\right)^2 - \frac{1}{2} m v_z^2 = \frac{1}{2} m \left(4uv_z + 4u^2\right) = 2mu(v_z + u)$$

Probabilities of Either Type of Collision

Denote the piston's lowest point as $z = 0$. In the below diagram two zones are drawn. The red zone represents the zone where particles will undergo a Type 1



collision, and the green zone represents the zone where particles will undergo a Type 2 collision.

We seek to find the widths of the zones Δz_1 and Δz_2. We do this by finding the positions of the particles such that they can reach the turning points of the piston in time before the piston changes direction. This gives

$$\frac{\Delta z_1 + 2a}{v_z} = \frac{2a}{u} \implies \Delta z_1 = 2a\left(\frac{v_z}{u} - 1\right)$$

$$\frac{\Delta z_1 + \Delta z_2}{v_z} = \frac{4a}{u} \implies \Delta z_2 = 2a\left(\frac{v_z}{u} + 1\right)$$

Since the particles are essentially distributed uniformly inside the cylinder, we can determine the probability of any given collision being a Type 1 collision as $p_1 = \frac{2a\left(\frac{v_z}{u}-1\right)}{\frac{4av_z}{u}} = \frac{1}{2} - \frac{u}{2v_z}$ and the probability of any given collision being a Type 2 collision as $p_2 = \frac{1}{2} + \frac{u}{2v_z}$. The combination of Type 2 collisions increasing the energy of the gas more than Type 1 collisions decrease it, as well as Type 2 collisions occurring more frequently, are the two factors that cause the rms speed of the particles to increase.

The Rate of RMS Speed Increase

The expected value of the energy change after 1 particle collision is

$$\begin{aligned}\delta E &= p_1 \Delta E_1 + p_2 \Delta E_2 \\ &= \left(\frac{1}{2} - \frac{u}{2v_z}\right)(-2mu(v_z - u)) + \left(\frac{1}{2} + \frac{u}{2v_z}\right)(2mu(v_z + u)) \\ &= 4mu^2\end{aligned}$$

Now let us sum this over one oscillation period. To do this, we first have to find the collision rate with the wall. Let us denote the system's temperature as $T = \frac{mv^2}{3k_B}$. If we let the number density of particles in the cylinder be n and the cylinder's base area be A, the particles with velocity v_z that hit the piston in time dt are contained in a cylinder with area A and height $v_z\ dt$ because the piston's surface is perfectly flat. Integrating this over the Boltzmann distribution for $v_z > 0$ gives us a total number of collisions in time t as

$$nAt\frac{\int_0^\infty v_z e^{-\frac{mv_z^2}{2k_BT}}dv_z}{\int_{-\infty}^\infty e^{-\frac{mv_z^2}{2k_BT}}dv_z} = nAt\sqrt{\frac{k_BT}{2\pi m}} = \sqrt{\frac{1}{6\pi}}nAvt$$

If we denote the total number of particles in the container as $N = nAH$, we can express the above number of collisions as $\sqrt{\frac{1}{6\pi}}\frac{Nvt}{H}$, and hence in one oscillation (which takes time $\frac{4a}{u}$), the number of collisions is

$$\sqrt{\frac{8}{3\pi}}\frac{Nva}{uH}$$

and so the expected energy change per oscillation is

$$\Delta E = \sqrt{\frac{8}{3\pi}}\frac{Nva}{uH}\left(4mu^2\right) = 8Nm\sqrt{\frac{2}{3\pi}}\frac{vau}{H}$$

Now note that as mentioned before, the velocities of the particles will be redistributed quickly. Hence we have $E = \frac{1}{2}Nmv^2 \implies \Delta E = Nmv\Delta v = 8Nm\sqrt{\frac{2}{3\pi}}\frac{vau}{H} \implies \Delta v = 8\sqrt{\frac{2}{3\pi}}\frac{au}{H}$. Interestingly, it does not depend on v. Now we find the number of oscillation periods r needed to double the rms speed ($\Delta v = v$):

$$8\sqrt{\frac{2}{3\pi}}\frac{au}{H}n = v \implies r = \sqrt{\frac{3\pi}{2}}\frac{Hv}{8au}$$

Noting that r is just $\frac{t}{\frac{4a}{u}}$ (the time elapsed divided by the oscillation period), we finally obtain

$$t = \left(\frac{4a}{u}\right)\left(\sqrt{\frac{3\pi}{2}}\frac{Hv}{8au}\right) = \sqrt{\frac{3\pi}{8}}\frac{Hv}{u^2}$$

Appendix

We can calculate the expected energy change per collision in the general case where v is not necessarily much larger than u via integrating over the Boltzmann distribution. Note that in this case, the partition function is $\int_0^\infty e^{-\frac{mv_z^2}{2k_BT}}dv_z$ because only particles with a positive z-velocity will collide with the piston.

Particles with speed $v_z \leq u$ will never catch up to the piston while it is moving up, hence they will always undergo a Type 2 collision. The expected energy change for these particles is

$$\delta e_1 = \frac{\int_0^u \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv_z^2}{2k_B T}} \left(2mu(v_z+u)\right)\ \mathrm{d}v_z}{\int_0^\infty e^{-\frac{mv_z^2}{2k_B T}} \mathrm{d}v_z}$$

$$= 4\sqrt{\frac{m}{2\pi k_B T}} \left(u^2 \mathrm{erf}\left(\sqrt{\frac{m}{2kT}}u\right)\sqrt{\frac{\pi}{2}mk_B T} + uk_B T\left(1 - e^{-\frac{mu^2}{2k_B T}}\right)\right)$$

where $\mathrm{erf}(x)$ is the error function, denoted by $\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$.

For particles with speed $v_z > u$, following the same logic as earlier with regard to the collision probabilities, the expected energy change for a Type 1 collision is

$$-\frac{\int_u^\infty \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv_z^2}{2k_B T}} \left(2mu(v_z-u)\right)\left(\frac{1}{2} - \frac{u}{2v_z}\right)\ \mathrm{d}v_z}{\int_0^\infty e^{-\frac{mv_z^2}{2k_B T}} \mathrm{d}v_z}$$

and the expected energy change for a Type 2 collision is

$$\frac{\int_u^\infty \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv_z^2}{2k_B T}} \left(2mu(v_z+u)\right)\left(\frac{1}{2} + \frac{u}{2v_z}\right)\ \mathrm{d}v_z}{\int_0^\infty e^{-\frac{mv_z^2}{2k_B T}} \mathrm{d}v_z}$$

Summing these two gives an additional contribution to the expected energy as

$$\delta e_2 = 4\sqrt{\frac{m}{2\pi k_B T}} \left(u^2\sqrt{2\pi m k_B T}\left(1 - \mathrm{erf}\left(\sqrt{\frac{m}{2k_B T}}u\right)\right)\right)$$

After some simplification (omitted for brevity), and remembering that $T = \frac{mv^2}{3k_B}$, the overall expected energy change is

$$\delta e = \delta e_1 + \delta e_2 = mu^2\left(4 - 2\ \mathrm{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{v}\right)\right) + 2\sqrt{\frac{2}{3\pi}}muv\left(1 - e^{-\frac{3u^2}{2v^2}}\right)$$

From here we proceed similarly to the original solution. The number of collisions in one oscillation period, as mentioned before, is $\sqrt{\frac{8}{3\pi}}\frac{Nva}{uH}$, so we have

$$\sqrt{\frac{8}{3\pi}}\frac{Nva\delta e}{uH} = Nmv\Delta v$$

$$\implies \Delta v = \sqrt{\frac{8}{3\pi}}\frac{a}{muH}\left(mu^2\left(4 - 2\ \mathrm{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{v}\right)\right) + 2muv\sqrt{\frac{2}{3\pi}}\left(1 - e^{-\frac{3u^2}{2v^2}}\right)\right)$$

$$= \left(\sqrt{\frac{8}{3\pi}}\frac{au}{H}\left(4 - 2\ \mathrm{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{v}\right)\right)\right) + \left(\frac{8a}{3\pi H}\left(1 - e^{-\frac{3u^2}{2v^2}}\right)\right)v$$

Now, let $A = \frac{8a}{3\pi H}\left(1 - e^{-\frac{3u^2}{2v^2}}\right)$ and $B = \sqrt{\frac{8}{3\pi}}\frac{au}{H}\left(4 - 2\ \mathrm{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{v}\right)\right)$. The solution to $\Delta v = Av + B$ is obtained via the integrating factor method, and is related to the number of oscillation periods m by

$$v' = \left(v + \frac{B}{A}\right)e^{Am} - \frac{B}{A}$$

where v' is the rms velocity at time $t > 0$. Setting $v' = 2v$ gives us

$$m = \frac{1}{A}\ln\left(\frac{2v + \frac{B}{A}}{v + \frac{B}{A}}\right)$$

$$\implies t = \frac{4a}{uA}\ln\left(\frac{2v + \frac{B}{A}}{v + \frac{B}{A}}\right)$$

$$= \frac{3\pi H}{2u\left(1 - e^{-\frac{3u^2}{2v^2}}\right)}\ln\left(\frac{2v + \sqrt{\frac{3\pi}{8}}\left(\frac{4-2\ \mathrm{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{v}\right)}{1-e^{-\frac{3u^2}{2v^2}}}\right)u}{v + \sqrt{\frac{3\pi}{8}}\left(\frac{4-2\ \mathrm{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{v}\right)}{1-e^{-\frac{3u^2}{2v^2}}}\right)u}\right)$$

What is left to do is show that this reduces to $\sqrt{\frac{3\pi}{8}}\frac{Hv}{u^2}$ in the limit $v \gg u$. First we note that for small x, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t \approx \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - t^2\right)\mathrm{d}t \approx \frac{2}{\sqrt{\pi}}x$ and $1 - e^{-x} \approx x$. This gives

$$t \approx \frac{3\pi H}{2u\left(\frac{3u^2}{2v^2}\right)}\ln\left(\frac{2v + \sqrt{\frac{3\pi}{8}}\left(\frac{2v^2}{3u}\right)\left(4 - 2\sqrt{\frac{6}{\pi}}\frac{u}{v}\right)}{v + \sqrt{\frac{3\pi}{8}}\left(\frac{2v^2}{3u}\right)\left(4 - 2\sqrt{\frac{6}{\pi}}\frac{u}{v}\right)}\right)$$

$$= \frac{\pi H v^2}{u^3}\ln\left(\frac{2 + \sqrt{\frac{2\pi}{3}}\left(2\frac{v}{u} - \sqrt{\frac{6}{\pi}}\right)}{1 + \sqrt{\frac{2\pi}{3}}\left(2\frac{v}{u} - \sqrt{\frac{6}{\pi}}\right)}\right) \approx \frac{\pi H v^2}{u^3}\ln\left(\frac{2 + \sqrt{\frac{8\pi}{3}}\frac{v}{u}}{1 + \sqrt{\frac{8\pi}{3}}\frac{v}{u}}\right)$$

Now we note that for large x, $\ln\left(\frac{2+x}{1+x}\right) = \ln\left(\frac{\frac{2}{x}+1}{\frac{1}{x}+1}\right) = \ln\left(1 + \frac{2}{x}\right) - \ln\left(1 + \frac{1}{x}\right)$ which reduces to $\frac{1}{x}$ via Taylor expansion of the logarithm ($\frac{1}{x}$ is small). This finally gives us

$$t \approx \frac{\pi H v^2}{u^3}\left(\sqrt{\frac{3}{8\pi}}\frac{u}{v}\right) = \sqrt{\frac{3\pi}{8}}\frac{Hv}{u^2}$$

Thus, we can conclude that simplifying the collision dynamics by assuming all the particles are moving at speed $v_z = \frac{v}{\sqrt{3}}$ is valid.