2025 Physics Cup Problem 3

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Solution

We first analyze the upward motion of the piston. Consider particles moving with a z-velocity of v_z, upwards. Since these particles move with a relative velocity of $v_z - u$ toward the piston, the number of such particles that hit the piston in a time interval dt will be

$$\frac{(v_z - u)dt}{H} N_0,$$

where N_0 is the total number of particles of gas in the cylinder. When an atom hits the piston, its relative velocity switches sign and it starts moving downwards with speed $v_z - 2u$. The change in energy of the atom is

$$\frac{1}{2}m(v_z - 2u)^2 - \frac{1}{2}mv_z^2 = 2mu(-v_z + u),$$

where m is the mass of each gas particle. Thus, the total amount of work done in time dt in a velocity range of $[v_z, v_z + dv_z]$ will be

$$2mu(-v_z + u) \cdot \frac{(v_z - u)dt}{H} N_0 \cdot \phi(v_z)dv_z,$$

where $\phi(v_z)$ is the distribution of z-velocities in the gas. Assuming that $\phi(v_z)$ remains constant throughout one cycle and equals the equilibrium distribution of velocities at temperature T (see Appendix A), we can integrate this expression over a half-period of $\tau_{1/2} = \frac{a}{u}$ to get

$$dE = -\frac{2maN_0}{H}(v_z - u)^2 \phi(v_z)dv_z.$$

Integrating over all positive velocities (see Appendix B), we get the change in energy of the gas over the first half-cycle is

$$\Delta E_1 = \frac{2maN_0}{H} \int_0^\infty (-v_z^2 + 2v_z u - u^2)\phi(v_z)dv_z.$$

The second half-cycle is similar. The change in energy of an atom moving upwards with z-velocity v_z will instead be

$$\frac{1}{2}m(v_z + 2u)^2 - \frac{1}{2}mv_z^2 = 2mu(v_z + u),$$

and the amount of work done in time dt to particles with velocity in a width dv_z is

$$2mu(v_z + u) \cdot \frac{(v_z + u)dt}{H} N_0 \cdot \phi(v_z)dv_z.$$

Similar to the first half-cycle, we integrate over time $\tau_{1/2} = \frac{a}{u}$ and over all positive velocities to get

$$\Delta E_2 = \frac{2maN_0}{H} \int_0^\infty (v_z^2 + 2v_z u + u^2)\phi(v_z)dv_z.$$

Summing over both half-cycles, the change in energy of the gas over one complete cycle is

$$\Delta E = \frac{2maN_0}{H} \int_0^\infty 4v_z u\phi(v_z)dv_z = \frac{8mauN_0}{H} \int_0^\infty v_z\phi(v_z)dv_z.$$

For a gas in equilibrium at temperature T, the z-velocities of the particles will satisfy a Maxwell-Boltzmann distribution:

$$\phi(v_z) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{mv_z^2}{2k_B T}\right).$$

Note that

$$\int_0^\infty xe^{-ax^2}dx = \frac{1}{2a}\int_0^\infty e^{-y}dy = \frac{1}{2a},$$

where we use the substitution $y = ax^2$. This gives

$$\begin{aligned}\int_0^\infty v_z\phi(v_z)dv_z &= \sqrt{\frac{m}{2\pi k_B T}}\int_0^\infty v_z \exp\left(-\frac{mv_z^2}{2k_B T}\right)dv_z \\ &= \sqrt{\frac{m}{2\pi k_B T}}\cdot\frac{k_B T}{m} = \sqrt{\frac{k_B T}{2\pi m}}.\end{aligned}$$

This gives

$$\Delta E = \frac{8mauN_0}{H}\cdot\sqrt{\frac{k_B T}{2\pi m}} = \frac{4auN_0}{H}\sqrt{\frac{2k_B T m}{\pi}}.$$

Since one cycle takes time $\tau = \frac{2a}{u}$, we have

$$\frac{dE}{dt} = \frac{2u^2 N_0}{H}\sqrt{\frac{2k_B T m}{\pi}} = \frac{3}{2}N_0 k_B\frac{dT}{dt},$$

where we have used the fact that the internal energy of a monoatomic gas is $\frac{3}{2}N_0 k_B T$. Separating variables, we have

$$\frac{1}{2\sqrt{T}}dT = \frac{2}{3}\frac{u^2}{H}\sqrt{\frac{2m}{\pi k_B}}dt.$$

The root-mean-square speed of a gas is given by $v = \sqrt{\frac{3k_B T}{m}}$, so we need the temperature to quadruple for the RMS speed to double. Hence,

$$\frac{2}{3}\frac{u^2}{H}\sqrt{\frac{2m}{\pi k_B}}\cdot t = \int_{T_0}^{4T_0}\frac{1}{2\sqrt{T}}dT = \sqrt{T_0} = \sqrt{\frac{m}{3k_B}}v,$$

so

$$t = \frac{1}{4}\sqrt{6\pi}\frac{Hv}{u^2}.$$

Appendix A: Deviation from equilibrium

Due to the interaction of the gas with the piston, the gas near the piston will cool down during the upward motion and heat up during the downward motion. In this section, we show that this effect is negligible and so that the true answer is consistent with the assumption that $\phi(v_z)$ remains constant throughout one cycle. Consider the first half-cycle. Since the particles must move λ on average before encountering another particle, the effect of the cooling will be less than that of an adiabatic cooling process occurring on a cylinder of height λ. Let the volume of this cylinder be V and the change in volume be αV. Since $a \ll \lambda$, $\alpha \ll 1$. Since $\gamma = \frac{5}{3}$ for a monoatomic gas, we have

$$T_0 V^{2/3} = T_1(V + \alpha V)^{2/3} \implies \frac{T_1}{T_0} = \left(\frac{1}{1+\alpha}\right)^{2/3} = 1 - \frac{2}{3}\alpha + O(\alpha^2).$$

This gives

$$W = \frac{3}{2}Nk_B\Delta T = -Nk_B T(\alpha + O(\alpha^2)).$$

Assuming that $\phi(v_z)$ is constant is equivalent to assuming that the pressure is constant throughout. In this case, we have

$$W = -p\alpha V = -Nk_BT\alpha.$$

The deviation in the amount of work done is on the order of

$$Nk_BT\alpha^2 \sim N_0k_BT \cdot \frac{\lambda}{H} \cdot \frac{a^2}{\lambda^2} \sim N_0k_BT \cdot \frac{a^2}{H\lambda},$$

which is quite small. In reality, the cooling effect will much smaller still, since the heat will diffuse throughout the entire cylinder. A similar argument can be made for the second half-cycle. Hence, the assumption that $\phi(v_z)$ remains constant is reasonably justified.

Appendix B: Ignoring small speeds

In the calculation of ΔE_1, all positive velocities were considered, when in reality, the integral should be taken from u to ∞. However, note that

$$\left|\int_0^u (u-v)^2\phi(v_z)dv_z\right| < \left|\int_0^u u^2\phi(v_z)dv_z\right| < u^3\sqrt{\frac{m}{2\pi k_BT}} \ll \sqrt{\frac{k_BT}{2\pi m}},$$

where the last inequality comes from $k_BT \sim mv^2$. Similarly, velocities between $-u$ and 0 were ignored in the calculation for ΔE_2, but have a similarly small effect.

Appendix C: Process is slow enough

From $\lambda \gg \frac{H^2}{vt}$, we get

$$t \gg \frac{H^2}{v\lambda} \gg \frac{H}{v}.$$

Thus, the assumption that the cylinder heats at a uniform rate throughout is justified.