Physics Cup Problem 3

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We first explain what the strong inequalities imply. The first strong inequality $H \gg \lambda \gg a$ implies that intermolecular collisions are significant when considering the entire gas (diffusive regime), such that we can use the Maxwell-Boltzmann distribution to describe the particles' velocities. Meanwhile, intermolecular collisions can be ignored near the piston (effusive regime), so we can treat the particles as mutually independent. The second strong inequality $\lambda \gg H^2/vt$ implies that the temperature of the gas is approximately uniform, as explained in Appendix 1. The final strong inequality $v \gg u$ allows us to ignore higher orders of u/v.

To reduce confusion we will now redefine v_{rms} as the rms speed of the gas molecules, and use v to represent speed of an arbitrary gas molecule. We define the z-axis as the axis of symmetry of the cylinder. As the piston moves upwards, we analyse how a gas particle of mass m with z-velocity v_z collides with it. In the piston's frame the particle's velocity is $v_z - u$, so the reflected velocity is $u - v_z$, which in the lab frame corresponds to a z-velocity of $2u - v_z$. The x and y-velocities are unchanged, so the change in energy of the particle is

$$\Delta E = \frac{1}{2}m((2u - v_z)^2 - v_z^2) = 2mu(u - v_z)$$

The rate at which particles with z-velocities between v_z and $v_z + dv_z$ (where $dv_z \ll v_z$) collide with the piston is

$$nA\,(v_z - u)f(v_z)\,dv_z$$

Here, n is the number density of gas particles, A is the surface area of the piston and $f(v_z)$ is the probability density of a particle having z-velocity v_z. Hence the rate of change of the total energy of the gas is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2nmuA\int_u^\infty (v_z - u)^2 f(v_z)\,dv_z$$

Similarly, when the piston moves downwards,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2nmuA\int_{-u}^\infty (v_z + u)^2 f(v_z)\,dv_z$$

Hence, on average, the rate of change of total energy is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2nmuA\frac{1}{2}\left(-\int_u^\infty (v_z - u)^2 f(v_z)\,dv_z + \int_{-u}^\infty (v_z + u)^2 f(v_z)\,dv_z\right)$$

We can simplify this by making approximations based on order-of-magnitude estimates. Specifically, if the lower bound of each integral is increased/decreased by $\sim u$, the integral will increase/decrease by $\sim u^3/v_{\text{rms}}$. Let us assume that this is an insignificant change and verify it later. Setting both lower bounds to 0:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = nmuA\left(\int_0^\infty 4v_z u\, f(v_z)\,dv_z\right)$$

Since the integral is on the order of $uv_{\text{rms}} \gg u^3/v_{\text{rms}}$, we confirm that increasing/decreasing the lower bounds by $\sim u$ will not significantly affect the value of the integral. Now, we can simplify this further by noting that

$$\int_0^\infty v_z f(v_z)\, dv_z = \frac{1}{2}\int_{-\infty}^\infty |v_z| f(v_z)\, dv_z = \frac{1}{2}\langle |v_z| \rangle$$

The average magnitude of the z-component of a random unit vector is $1/2$, so $\langle |v_z| \rangle = \langle v \rangle / 2$, as the direction of the velocity of each particle is completely random. An interesting proof of this is in Appendix 2.

$$\therefore \frac{\mathrm{d}E}{\mathrm{d}t} = 2nmu^2 A \langle |v_z| \rangle = nmu^2 A \langle v \rangle$$

The mean speed is a well-known quantity, so we just use that:

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}} = \sqrt{\frac{8}{3\pi}} v_{\text{rms}} \implies \frac{\mathrm{d}E}{\mathrm{d}t} = nmu^2 A \sqrt{\frac{8}{3\pi}} v_{\text{rms}}$$

Using $E = Nmv_{\text{rms}}^2/2$ and $n = N/AH$ where N is the total number of gas particles,

$$\frac{\mathrm{d}v_{\text{rms}}^2}{\mathrm{d}t} = 2\sqrt{\frac{8}{3\pi}} \frac{u^2}{H} v_{\text{rms}}$$

Using the chain rule,

$$\frac{\mathrm{d}v_{\text{rms}}^2}{\mathrm{d}t} = \frac{\mathrm{d}v_{\text{rms}}^2}{\mathrm{d}v_{\text{rms}}} \frac{\mathrm{d}v_{\text{rms}}}{\mathrm{d}t} = 2v_{\text{rms}} \frac{\mathrm{d}v_{\text{rms}}}{\mathrm{d}t} = 2\sqrt{\frac{8}{3\pi}} \frac{u^2}{H} v_{\text{rms}}$$

$$\therefore \frac{\mathrm{d}v_{\text{rms}}}{\mathrm{d}t} = \sqrt{\frac{8}{3\pi}} \frac{u^2}{H}$$

The RMS speed increases at a constant rate, so the time taken for the RMS speed to increase by v is

$$\boxed{t = \frac{\sqrt{6\pi}}{4} \frac{vH}{u^2}}$$

Appendix 1: Second strong inequality $\lambda \gg H^2/vt$

We can estimate the timescale of heat propagation by creating a simplified model of heat conduction (diffusion) in a gas. The added energy received by a gas particle near the piston is transferred from one particle to another. In between collisions, particles have a random speed and direction. This describes a random walk.

The variance of the distance the added energy "travels" can be found by adding the variances of each random step. In each random step (i.e. collision), the variance of the displacement is on the order of λ^2. So after N steps, the variance of the total displacement is $\sim N\lambda^2$, which means that the expected distance travelled is $\sim \sqrt{N}\lambda$. For heat to be transferred to the bottom of the cylinder, $H \sim \sqrt{N}\lambda \implies N \sim H^2/\lambda^2$. The distance between collisions is $\sim \lambda$ and the particles travel at speed $\sim v$ so the time taken for heat to propagate is $t_H \sim N\lambda/v \sim H^2/\lambda v$.

$$\lambda \gg H^2/vt \implies t \gg t_H$$

Hence, the second strong inequality implies that the timescale of heat conduction is much lower than the timescale of the temperature rise, so that we can assume that the temperature of the gas is roughly uniform at any time.

Appendix 2: Proof that $\langle |v_z| \rangle = \langle v \rangle /2$

As mentioned earlier, this is equivalent to proving that the average magnitude of the z-component of a random unit vector is 1/2. If we treat each unit vector as the displacement vector of an infinitesimal mass, this question becomes one of finding the height of the CM of a uniform hemispherical shell with unit radius. We will let the hemispherical surface be S.

$$h_{\text{CM}} = \frac{1}{S} \int_S z\, dS = \frac{1}{2\pi} \int_S \hat{\mathbf{n}} \cdot \mathbf{dS}$$

The integral of the normal vector over any two surfaces with the same bounds is equal, so we can replace the hemisphere with a circle, which has an area of π.

$$\therefore h_{\text{CM}} = \frac{\pi}{2\pi} = \frac{1}{2}$$