

Physics cup 2026 Problem 2

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1 Understanding the System

To begin, notice there is $\frac{2\pi}{k}$ periodicity in $v_x(x)$. Also, molecules at $0 < x < \frac{\pi}{k}$ will have positive velocity, and molecules at $\frac{\pi}{k} < x < \frac{2\pi}{k}$ will have negative velocity. Thus, the behavior on all segments of the form $\left[\frac{2\pi n}{k}, \frac{2\pi(n+1)}{k}\right]$ is identical, so it will be enough to consider only $\left[0, \frac{2\pi}{k}\right]$. Notice at the start the macroscopic velocity everywhere is toward the point $x_0 = \frac{\pi}{k}$. From this we understand that this is a special point, and it would be nicer to work around it. Hence, we define

$$x \rightarrow x - \frac{\pi}{k}$$
$$v_x(x) = -v_0 \sin(kx)$$

And now the problem is centered around $x = 0$.

Denote the distribution of the velocity in the x-direction at location x and time t by:

$$f_{x,t}(v_x)$$

This is independent of y, z and uniform for v_y, v_z since the system is initially homogenic in these variables.

In the limit of $n \rightarrow 0, T \rightarrow 0$, meaning in the limit of no collisions and molecule velocities given exactly by $v_x(x)$, A multi-stream flow will arise when molecules with different initial velocities meet at a point. The first point this will happen at is of course $x = 0$. In the general case, as time goes on, more and more collisions will happen, which change molecule velocities almost arbitrarily (while respecting conservation laws), so the distribution will lose it's shape from the collision-less limit. As a result, in order to figure out if a multi-stream arises, we should only consider the points where it would happen first, so only at $x = 0$.

We will proceed with the solution in two steps:

1. Find the time evolution of $f_{0,t}(v_x)$ assuming no collisions and figure out when a multi-stream arises in this case.
2. Find the critical order of magnitude of initial molecule location, x_0 , for which too many collisions will ruin the distribution shape derived in the first step.

2 The Collision-less limit

In the linear approximation, $v_x(x) = -v_0 kx$. We assume that $v_0 \gg \sqrt{\frac{k_B T}{m}}$, so the variation in initial velocity is small. Thus, in this approximation, all molecules will reach $x = 0$ at time $t^* = \frac{1}{v_0 k}$. So, in order to differentiate between two initial

locations, we have to consider the next order. Now, molecules starting farther away will reach $x = 0$ after a slightly longer time. In addition, the variance in initial velocity will slightly change the time of arrival.

Assuming there are no collisions, the probability of finding a molecule of speed v at a point x and time t is the same as finding it $x = x_0 - vt, t = 0$. The number of molecules with $x_0 < x < x_0 + dx$, $v_0 < v < v_0 + \delta v$ per unit area in the yz plane at time t is:

$$N(x_0, v_0) = f_{x_0, t}(v_0) \cdot dx dv$$

But in the collision-less limit, since the the distribution is uniform in space at $t = 0$, we also have:

$$N = f_{x_0 - v_0 t, 0}(v_0) \cdot [(x_0 + dx - v_0 t) - (x_0 - v_0 t)] dv = f_{x_0 - v_0 t, 0}(v_0) \cdot dx dv$$

$$\Rightarrow f_{x_0, t}(v_0) = f_{x_0 - v_0 t, 0}(v_0)$$

This is of course up to a multiplicative constant, since f should be normalized. This constant doesn't matter though, since we are looking for maxima twice as high as the minima, and this doesn't change when scaling by a constant factor.

When there is no macroscopic velocity, the velocity distribution is of course given by the Maxwell-Boltzmann (MB) distribution. Since there is some macroscopic velocity, the distribution will still be given by MB, but centered around that macroscopic velocity:

$$f_{x, t}(v) = f_{x - vt, 0}(v) = f_{MB}(v - v_x(x - vt))$$

The MB distribution is:

$$f_{MB}(v) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv^2}{2k_B T}}$$

Although we do know this exact form, we won't need it, since we aren't interested in the prefactor. It is sufficient to know that it falls off with characteristic velocity $\sqrt{\frac{k_B T}{m}}$.

We will now use the previous result to figure out when a multi-stream flow arises at $x = 0$. This means we are interested in finding the maxima and minima of $f_{0, t}(v)$. As explained earlier, molecules with initial $x \ll \frac{1}{k}$ will reach $x = 0$ after a time slightly longer than t^* which we will denote by $t = t^* + \delta t$. Working under the approximations of particles, coming from $x \ll \frac{1}{k}$ we immediately have

$$\frac{\delta t}{t^*} \ll 1, kvt = kv t^* \ll 1$$

At the end, will explicitly make sure that the approximation is okay. Recall that $t^* = \frac{1}{v_0 k}$ and calculate to leading order in the small variables:

$$\begin{aligned} v_x(-vt) &= -v_0 \sin(-kvt) = v_0 \sin(kvt) \approx v_0 \left(kvt + \frac{(kvt)^3}{6} \right) = v_0 \left(kv(t^* + \delta t) - \frac{(kv(t^* + \delta t))^3}{6} \right) \\ &\approx v + kv_0 v \cdot \delta t - \frac{v_0 k^3 v^3 t^{*3}}{6} = v + kv_0 v \cdot \delta t - \frac{v^3}{6v_0^2} \\ &\Rightarrow f_{0, t}(v) = f_{MB} \left(\frac{v^3}{6v_0^2} - kv_0 v \cdot \delta t \right) \end{aligned}$$

The maxima correspond to the argument of f_{MB} being 0 and the minima correspond to it being largest in absolute value. Calculating exactly v_{max}, v_{min} and the δt corresponding to the first time that there is a multi-stream at $x = 0$ is not difficult, but not necessary. It's clear that there are three maxima, one at $v = 0$, and two symmetric ones at:

$$v_{max}^2 = 6kv_0^3 \cdot \delta t$$

the minima in between are of the same magnitude as these maxima:

$$v_{min} \sim v_{max}$$

We want the maximum to be twice larger than the minimum:

$$\frac{1}{2} \stackrel{!}{=} \frac{f_{0,t}(v_{min})}{f_{0,t}(v_{max})} \sim \frac{f_{MB}\left(\frac{v_{min}^3}{v_0^3}\right)}{f_{MB}(0)}$$

Since the distribution falls off with characteristic velocity $\sqrt{\frac{k_B T}{m}}$, requiring it to drop to half maximum implies that the corresponding velocity must be of order $\sqrt{\frac{k_B T}{m}}$:

$$\left| \frac{v_{min}^3}{v_0^3} \right| \sim \sqrt{\frac{k_B T}{m}}$$

Of course, velocities that interest us - meaning that they form the interesting part of the distribution are of the same magnitude as $v_{min,max}$:

$$\left| \frac{v^3}{v_0^3} \right| \sim \sqrt{\frac{k_B T}{m}}$$

and the corresponding x values for the molecules' initial location:

$$\begin{aligned} x &= vt \approx vt^* \\ \Rightarrow x^3 &\approx v^3 t^{*3} \sim \frac{v_0^2 \sqrt{\frac{k_B T}{m}}}{v_0^3 k^3} = \frac{\sqrt{\frac{k_B T}{m}}}{v_0 k^3} \end{aligned}$$

Now we can justify our approximation explicitly. Since we are working under the assumption that $v_0 \gg \sqrt{\frac{k_B T}{m}}$, we get

$$(kx)^3 = \frac{\sqrt{\frac{k_B T}{m}}}{v_0} \ll 1$$

3 Collisions

To understand why collisions are important, one first needs to understand how they work. The exact distribution of how the molecules are scattered is determined by the nature of the interaction between them and the differential cross section

derived from it. In the case of elastic scattering of two spheres of radius r , it is well known that in the CM frame:

$$\frac{d\sigma}{d\Omega} = r^2, \sigma = 4\pi r^2$$

All this means is that the total effective (in this case also the actual) scattering area is $4\pi r^2 = \pi (2r)^2$ and that it is uniform in the scattering direction. Note that this is very different from 1-dimensional elastic collision between two identical spheres since in this case the spheres swap velocities and effectively nothing happens.

Note that this is true in the CM frame, so the molecules' average velocity stays the same and their difference changes to an arbitrary direction. This means that the velocities of the colliding molecules go to arbitrary, opposite, points on the sphere in velocity space whose diameter is the two initial velocities. This means that collisions of two molecules with similar velocities are not interesting, meaning that collisions due to thermal variations, of molecules that started with similar x are not interesting since $v_0 \gg \sqrt{\frac{k_B T}{m}}$. Hence, the only collisions that need to be considered are those between molecules that started with different velocities and at a different location along the x-axis. As we have already seen, these collisions start happening at $t = t^*$.

Collisions will have two effects on the distribution we calculated earlier:

1. molecules that we considered earlier to reach $x = 0$ at $t = t^* + \delta t$ won't reach it due to having collided.
2. molecules that weren't supposed to be at $x = 0$ at $t = t^* + \delta t$ will reach it due to colliding and acquiring a new velocity. In addition, this new velocity is more likely to be smaller in the x-direction since some of it will go into the y,z directions (going back to the sphere in velocity space, it is "widest" at the average of the two initial velocities, which is usually $v_x = 0$).

both of these effects are roughly of the same order, so it is enough to require that a molecule that reaches $x = 0$ at $t = t^* + \delta t$ will have a probability considerably smaller than 1 to collide before reaching $x = 0$. This means that it has to be less than some constant which is of order 1, which ensures that the noise introduced by the collisions doesn't affect the distribution enough.

Consider a cross section A . A molecule starting at $x = x_0$ and reaching $x = 0$ will meet all the molecules starting between $-x_0$ and 0, because the molecules starting at $-x_0$ also exactly reach 0 and molecules closer than them to 0 will get slightly further. so the number of molecules that the molecule goes by is roughly:

$$N = n \cdot A \cdot x_0$$

Since we require the probability of collision to be considerably less than 1, it is enough to require that the total scattering area is considerably smaller than the total area considered, A . since we don't care about the prefactor, it is enough to require that their ratio is less than 1:

$$p = \frac{A_{tot}}{A} = \frac{N \cdot \pi r^2}{A} = \pi n x_0 r^2 < 1$$

Another way to reach this is understanding that at the start of collisions, at time $t = t^*$, all the molecules that started between $-x_0$ and x_0 are now confined to $-x_1, x_1$ where x_1 is the location of molecules starting at $x = x_0$ at $t = t^*$. thus $n_1 \approx n \cdot \frac{x_0}{x_1}$ and the mean free path is

$$\lambda = \frac{1}{n' \cdot \pi r^2} = \frac{x_1}{\pi n \cdot x_0 r^2}$$

and we require

$$\lambda > x_1 \Rightarrow \pi n x_0 r^2 < 1$$

Now we have an inequality for x_0 , which the molecules making up the distribution from earlier have to satisfy, in order for the derivation of the collision-less limit to have meaning. We saw that

$$x_0^3 \sim \frac{\sqrt{\frac{k_B T}{m}}}{v_0 k^3}$$

plugging this in and dropping the π , we get:

$$1 > n^3 r^6 x_0^3 \sim n^3 r^6 \cdot \frac{\sqrt{\frac{k_B T}{m}}}{v_0 k^3}$$

$$\boxed{v_0 > \frac{n^3 r^6}{v_0 k^3} \cdot \sqrt{\frac{k_B T}{m}}}$$

As a sanity check - the units make sense and in the limits, $n \rightarrow 0, T \rightarrow 0$, we indeed get that a multi-stream arises, as discussed at the beginning.