

Physics Cup problem 1 - Trapped block

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1 Problem statement

We consider block squished between 2 rigid plates (fig. 1). At the beginning block velocity $\mathbf{u}_0$ is perpendicular to upper plate velocity $\mathbf{v}$. We consider dry friction between block and plates with friction constant μ . The system is stationary in $\hat{\mathbf{z}}$ direction, thus vertical forces acting on the block from the plates must have same magnitude F . Last the block has mass m . Then it's equation of motion is:

$$m \frac{d\mathbf{u}}{dt} = -\mu F \left(\frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} + \frac{\mathbf{u}}{|\mathbf{u}|} \right) \quad (1)$$

Where dry friction is always opposite to relative velocity of brick. RHS of this equation is just sum of friction forces acting on the brick. We can define parameter $\alpha = \frac{\mu F}{m}$, it's dimension is $\frac{m}{s^2}$. We can also connect it's value to a_0 given in the problem with:

$$a_0^2 = \dot{\mathbf{u}}^2(0) = \alpha^2 \left(\frac{\mathbf{u}_0 - \mathbf{v}}{|\mathbf{u}_0 - \mathbf{v}|} + \frac{\mathbf{u}_0}{u_0} \right)^2 = \alpha^2 \left(2 + 2 \frac{u_0^2 - \mathbf{u}_0 \cdot \mathbf{v}}{u_0 |\mathbf{u}_0 - \mathbf{v}|} \right)$$

Now we can observe, that for the initial conditions $\mathbf{u}_0 \cdot \mathbf{v} = 0$. Then we can finally obtain:

$$a_0 = \alpha \sqrt{2 + \frac{2u_0}{\sqrt{u_0^2 + v^2}}}$$

What is asked in the problem is displacement of the block in the limit of the large times:

$$l_y = \lim_{T \rightarrow \infty} \int_0^T u_y(t) dt = \int_0^\infty u_y(t) dt \quad (2)$$

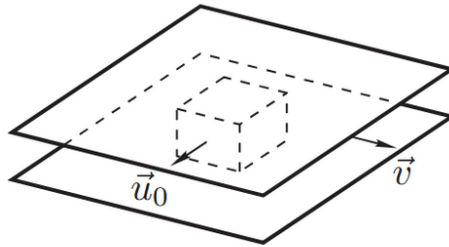


Figure 1: Block squished between plates. Figure from problem statement for Physics Cup website [1].

2 Solution strategy and prerequisites

In our problem trajectory in velocity space is defined by parameter t (time). Then we have values $u_x(t)$ and $u_y(t)$. The trick is to find some integral of motion and express trajectory with more suitable parameter. This parameter needs to be monotonic in time, so we would be able to re-parametrize integral with it. Our quest to finding such a parameter start with looking at the equilibrium state of equations of motion (eq. 1).

2.1 Equilibrium state

In the equilibrium $\dot{\mathbf{u}} = 0$, thus we have equation defining equilibrium state $\mathbf{u}_\infty$:

$$0 = \frac{\mathbf{u}_\infty - \mathbf{v}}{|\mathbf{u}_\infty - \mathbf{v}|} + \frac{\mathbf{u}_\infty}{|\mathbf{u}_\infty|}$$

We can easily rewrite this equation as:

$$\mathbf{v} = \mathbf{u}_\infty \left(1 + \frac{|\mathbf{u}_\infty - \mathbf{v}|}{|\mathbf{u}_\infty|} \right)$$

So $\mathbf{u}_\infty$ must be parallel to $\mathbf{v}$. Then equation is fulfilled for any $u_\infty = |\mathbf{u}_\infty| \leq v$. So equilibrium velocity (attractor of our ODE) is:

$$\mathbf{u}_\infty = u_\infty \frac{\mathbf{v}}{v}, \quad 0 < u_\infty \leq v$$

There is no single attractor by whole continuum of them. With this in mind we can generate easily verify problem reformulation ideas. They need to necessary have the same equilibrium.

2.2 Problem reformulation - the trick

We can now observe, that:

$$\frac{\mathbf{u}}{u} = \nabla_{\mathbf{u}} u$$

This is known fact widely used in electrostatics and hydrodynamics. With this trick we can reformulate eq. 1:

$$\dot{\mathbf{u}} = -\alpha \nabla_{\mathbf{u}} (|\mathbf{u} - \mathbf{v}| + u)$$

This is gradient descent equation (as $\alpha > 0$)! We can define function $f(\mathbf{u}) = |\mathbf{u} - \mathbf{v}| + u$, then it's evolution is:

$$\frac{df}{dt} = \dot{\mathbf{u}} \cdot \nabla_{\mathbf{u}} f = -\alpha |\nabla_{\mathbf{u}} f|^2$$

In this problem we are effectively minimizing $f(\mathbf{u})$. We can see that $f(\mathbf{u})$ is minimal when $\mathbf{u}$ is parallel to $\mathbf{v}$ and has smaller modulus - the same equilibrium as original problem. We can also observe that:

$$\frac{1}{\alpha} \frac{du_x}{dt} = -\partial_{u_x} f = \frac{v - u_x}{\sqrt{(v - u_x)^2 + u_y^2}} - \frac{u_x}{\sqrt{u_x^2 + u_y^2}} \quad (3)$$

$$\frac{1}{\alpha} \frac{du_y}{dt} = -\partial_{u_y} f = \frac{-u_y}{\sqrt{(v - u_x)^2 + u_y^2}} - \frac{u_y}{\sqrt{u_x^2 + u_y^2}} \quad (4)$$

We can rewrite equation 4 as:

$$\frac{du_y}{dt} = -u_y \left(\frac{\alpha}{\sqrt{(v - u_x)^2 + u_y^2}} + \frac{\alpha}{\sqrt{u_x^2 + u_y^2}} \right)$$

For any $u_0 > 0$ term multiplying $-u_y$ is positive and finite in the limit $t \rightarrow \infty$ (equilibrium condition), thus u_y monotonically decays in time from $u_y(0) = u_0$ to $u_y(\infty) = 0$. In conclusion u_y is a great candidate for the reparametrization! We only need to express u_x as a function of u_y . For that we need some integral of motion.

2.3 Integral of motion

By playing around we can observe that:

$$\begin{aligned} \frac{d}{dt} (|\mathbf{u} - \mathbf{v}| - u) &= \dot{\mathbf{u}} \cdot \nabla_{\mathbf{u}} (|\mathbf{u} - \mathbf{v}| - u) = \nabla_{\mathbf{u}} (|\mathbf{u} - \mathbf{v}| + u) \cdot \nabla_{\mathbf{u}} (|\mathbf{u} - \mathbf{v}| - u) \\ &= |\nabla_{\mathbf{u}} |\mathbf{u} - \mathbf{v}||^2 - |\nabla_{\mathbf{u}} u|^2 = 1 - 1 = 0 \end{aligned}$$

So we have our integral of motion: $|\mathbf{u} - \mathbf{v}| - u$. From it we can express u_x as a function of u_y . First we need to equate general value of $|\mathbf{u} - \mathbf{v}| - u$ to it's initial value (as it is constant in time).

$$\sqrt{(v - u_x)^2 + u_y^2} - \sqrt{u_x^2 + u_y^2} = \sqrt{v^2 + u_0^2} - u_0 = C$$

There we defined useful constant C . We can solve such an equation for u_x . It will be quadratic so we get two roots:

$$u_x = \frac{v}{2} \pm \frac{C}{2} \sqrt{1 + \frac{4u_y^2}{v^2 - C^2}} \quad (5)$$

We need to determine which is physically correct solution and it needs to fulfill boundary condition $u_x(0) = 0$, for $u_y(0) = u_0$, thus the correct is one with "-" sign. Armed with this knowledge we can also find an expression for $\sqrt{u_x^2 + u_y^2}$ as a function of u_y :

$$\sqrt{u_x^2 + u_y^2} = -\frac{C}{2} + \frac{v}{2\Delta} \sqrt{4u_y^2 + \Delta^2} = -\frac{C}{2} + R(u_y) \quad (6)$$

Here we defined another helpful constant $\Delta^2 = v^2 - C^2 = 2u_0C$. It is easy to check that $\Delta^2 > 0$. Knowing all that we can proceed to solve question asked in the problem - evaluating displacement l_y .

3 Solution

We now need to transform equation for l_y (eq. 2) to our new parametrization:

$$l_y = \int_0^\infty u_y(t) dt = \int_{u_0}^0 u_y \frac{du_y}{\dot{u}_y}$$

We can plug here equation 4 to obtain:

$$\alpha l_y = \int_0^{u_0} u_y \frac{du_y}{\frac{u_y}{\sqrt{(v-u_x)^2 + u_y^2}} + \frac{u_y}{\sqrt{u_x^2 + u_y^2}}}$$

Apart from obvious simplification we can use the fact that $\sqrt{(v - u_x)^2 + u_y^2} = C + \sqrt{u_x^2 + u_y^2}$ to obtain:

$$\alpha l_y = \int_0^{u_0} \frac{du_y}{\frac{1}{C + \sqrt{u_x^2 + u_y^2}} + \frac{1}{\sqrt{u_x^2 + u_y^2}}}$$

Now we can use derived value for $-\frac{C}{2} + R(u_y) = \sqrt{u_x^2 + u_y^2}$ (eq. 6) to simplify solution. There $R(u_y) = \frac{v}{2\Delta} \sqrt{4u_y^2 + \Delta^2}$ In addition I will rename the integration variable $u_y \rightarrow y$:

$$\alpha l_y = \int_0^{u_0} \frac{R^2(y) - \frac{C^2}{4}}{2R(y)} dy = \frac{1}{2} \int_0^{u_0} \left[R(y) - \frac{C^2}{4R(y)} \right] dy$$

So essentially we need to evaluate 2 integrals. After the change of variables $2y = x$ expression for displacement becomes:

$$\alpha l_y = \frac{v}{8\Delta} \int_0^{2u_0} \sqrt{x^2 + \Delta^2} dx - \frac{\Delta C^2}{8v} \int_0^{2u_0} \frac{dx}{\sqrt{x^2 + \Delta^2}}$$

Both of them are standard integrals from tables or one can do them by substituting area hyperbolic functions:

$$\begin{aligned} \int \sqrt{x^2 + \Delta^2} dx &= \frac{1}{2} x \sqrt{\Delta^2 + x^2} + \frac{1}{2} \Delta^2 \tanh^{-1} \left(\frac{x}{\sqrt{\Delta^2 + x^2}} \right) \\ \int \frac{dx}{\sqrt{x^2 + \Delta^2}} &= \tanh^{-1} \left(\frac{x}{\sqrt{\Delta^2 + x^2}} \right) \end{aligned}$$

Values at $x = 0$ give no contribution. Evaluated expression, after using logarithmic form $\tanh^{-1}(x) = \frac{1}{2} \log \frac{1+x}{1-x}$ and introducing common denominator in logarithm, simplifies to:

$$\alpha l_y(u_0, v) = \frac{\Delta}{16v} (2C^2 - v^2) \log \left(\frac{\sqrt{\Delta^2 + 4u_0^2} - 2u_0}{\Delta} \right) + \frac{u_0 v \sqrt{\Delta^2 + 4u_0^2}}{8\Delta} \quad (7)$$

This is answer I submitted initially.

3.1 Further simplification

Equation 7 can be further simplified when we observe that $\sqrt{\Delta^2 + 4u_0^2} = \frac{2\Delta}{v} R(u_0) = \frac{2\Delta}{v} \left(\frac{C}{2} + u_0 \right)$. This can be verified with the definition of $R(u_y)$ (eq. 6) or by long algebraic computation. Finally we arrive at the final solution:

$$\alpha l_y(u_0, v) = \frac{\Delta}{16v} (2C^2 - v^2) \log \left(\frac{C}{v} + \frac{2u_0}{v} + \frac{2u_0}{\Delta} \right) + \frac{u_0}{8} (C + 2u_0) \quad (8)$$

Where:

- $\alpha = \frac{a_0}{\sqrt{2 + \frac{2u_0}{\sqrt{u_0^2 + v^2}}}}$
- $C = \sqrt{u_0^2 + v^2} - u_0$
- $\Delta = \sqrt{v^2 - C^2} = \sqrt{2u_0 C}$

With numerical answer (rounded to 4° digit):

$$l_y \left(u_0 = 1 \frac{m}{s}, v_0 = 1 \frac{m}{s}, a_0 = 1 \frac{m}{s^2} \right) = 0.6631m$$

References

- [1] *Problem statement on physics cup website.* <https://physicscup.ee/physics-cup-taltech-2026-problem-1/>.