

P2 - Multi-stream gas flow

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1 Some physical considerations

First of all, we have to understand where and when we should observe the multi-stream flow. The initial state is highly not homogeneous if we consider velocities, and we can expect the gas to reach in a short time a Maxwell-like distribution.

Let us focus on the first moments. Since macroscopic velocities are given by $v_x(x) = v_0 \sin(kx)$ (we suppose without loss of generality $k, v_0 > 0$), we can expect all the particles whose position along x -axis is between 0 and $\frac{2\pi}{k}$ to go toward the plane $x = \pi$, and, in general, we can expect all the particles whose position satisfies $\frac{2z\pi}{k} < x < \frac{2(z+1)\pi}{k}$ to go toward the plane $x = \frac{(2z+1)\pi}{k}$, $z \in \mathbb{Z}$.

Thus, we shall expect a growing density in all the planes $x = \frac{(2z+1)\pi}{k}$ during the first moments. After a certain moment, however, the density and consequently the pressure in these planes will be so high that we should observe an expansion and a decreasing density in the planes and in their neighbourhood. First of all, the assumption $v_0 \gg \sqrt{\frac{k_b T}{m}}$ means here that we can, only for the moment, neglect random oscillations of the velocities of the particles due to thermal effects. Now, without loss of generality let us suppose $0 < x < \frac{2\pi}{k}$ until the end of this section. It follows that time τ needed by the particles between $x - dx$ and x to reach the particles between x and $x + dx$ can be calculated as the space they have to travel divided by the relative speed, i.e.

$$\tau = \frac{dx}{kv_0 \cos(kx)dx} = \frac{1}{kv_0 \cos(kx)}$$

It is interesting that time τ_0 needed by the same particles to reach plane $x = \frac{\pi}{k}$, neglecting all the interactions, is

$$\tau_0 = \frac{\pi - kx}{kv_0 \sin(kx)}$$

We can verify that $\tau_0 < \tau$ for every $0 < x < \frac{\pi}{k}$, defining $y := \frac{\pi}{k} - x$ ($-\frac{\pi}{k} < y < 0$):

$$\frac{y}{v_0 \sin(ky)} < \frac{1}{kv_0 \cos(ky)}$$

is equivalent to

$$\frac{y^2}{v_0^2 \sin^2(ky)} - \frac{1}{k^2 v_0^2 \cos^2(ky)} = \frac{k^2 x^2 \cos^2(ky) - \sin^2(ky)}{k^2 v_0^2 \sin^2(ky) \cos^2(ky)} < 0$$

Since, if $0 < y < \frac{\pi}{k}$

$$\frac{d}{d(ky)} (ky \cos(ky) - \sin(ky)) = -ky \sin(ky) < 0$$

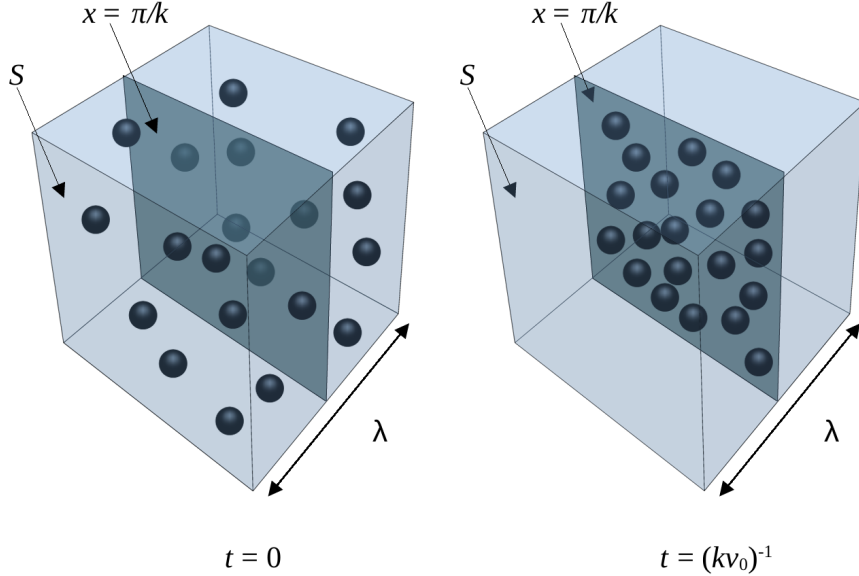
$$\sin(kx), kx \cos(kx) > 0$$

and since, in $y = 0$, $ky \cos(ky) - \sin(ky) = 0$, we have $ky \cos(ky) < \sin(ky) \Rightarrow k^2 y^2 \cos^2(ky) - \sin^2(ky) < 0 \forall y \in (0, \frac{\pi}{k})$. This result can naturally be extended $\forall y \in \mathbb{R}$ and consequently $\forall x \in \mathbb{R}$. It then immediately follow that $\tau > \tau_0$, for each particle.

This means that, neglecting statistical oscillations, the first collisions with particles having a different macroscopic speed will happen with the particles near $x = \frac{\pi}{k}$, and not with the particles in the

neighbourhood. Thus, macroscopic velocities are constant until they reach a region near $x = \frac{\pi}{k}$ (what we shall intend with "near" will be analyzed later).

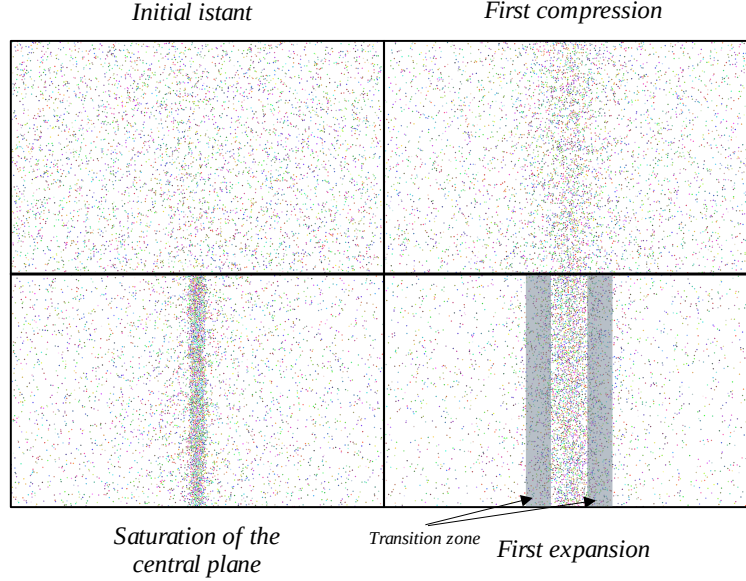
Let us now estimate the density at $x = \frac{\pi}{k}$. Since macroscopic velocities of the particles grows linearly near this plane, we can expect a very high density after a time $\tau_0 = \lim_{x \rightarrow \frac{\pi}{k}} \frac{\pi - kx}{v_0 \sin(kx)} = \frac{1}{kv_0}$. However, density cannot grow indefinitely. We see in the following figure a parallelepiped of section S and height λ ; at the middle there is the plane $x = \frac{\pi}{k}$.



If we suppose $\lambda \ll \frac{1}{k}$, then macroscopic velocities of the particles in this parallelepiped can be described by the linear function $v_x = kv_0 \left(\frac{\pi}{k} - x \right)$. This means that all the particles in the parallelepiped will be at $x = \frac{\pi}{k}$ after time $\tau_0 = \frac{1}{kv_0}$, but only if $n\lambda S \sim \frac{S}{r^2}$. In fact, the number of particles in the surface S cannot be greater than $\frac{S}{r^2}$ (within a numerical prefactor). If $n\lambda S$ is greater than this maximum number, we will consequently observe a very high number of collisions in a very short time, as the density near $x = \frac{\pi}{k}$ will be very great. Otherwise, if $n\lambda \sim \frac{1}{r^2}$, we shall expect only a few collisions, and thus we shall expect no significative change in the speed of each particle in the parallelepiped. In conclusion, it is worth noting that λ corresponds to the free mean path of a particle of radius r in a gas of constant density n .

We now want to study what will happen if $\lambda \ll \frac{1}{k}$. We have already seen that in correspondence of $x = \frac{\pi}{k}$ the density will reach its maximum and we will observe a very high number of collisions in a short time. This means that, after the saturation of the central plane, in a very short time velocities of the particles near $x = \frac{\pi}{k}$ will be described by a distribution similar to Maxwellian one, with a temperature T_s proportional to the mean square speed of the particles which have reached this region. Then, since this moment and for the rest of the evolution of the gas we are sure we cannot observe a multi-stream flow here. Thus, after the saturation of the central plane, if we see a multi-stream flow we will see it in another region; in particular, since the density near the central plane is very great, also the pressure will grow fastly, and we see an expansion of the particles near $x = \frac{\pi}{k}$. Since we are looking for a multi-stream flow, we shall expect it in the transition zone between expanding particles in the central region and particles which have still their initial macroscopic speed. It is very hard to give an exact description of what happens in this transition region; however, we can expect expanding particles to have a velocity distribution function similar to Maxwellian one, scaled along the v_x axis proportionally to $\sqrt{T_s} \propto v_0$. In fact, since $v_0 \gg \sqrt{\frac{k_b T}{m}}$, if we double v_0 we see approximately the same evolution, but with all the velocities doubled. Thus, since the velocity distribution of the expanding particles is scaled proportionally to v_0 , and since the maximum of the velocity distribution of the incoming particles is also scaled proportionally to v_0 , a multi-stream flow arises in this moment and

in this region with a certain condition for T , not for v_0 . A different value of v_0 will not, in fact, alter the relative ratios of minima and maxima, but just their corresponding values of v_x .



All this considerations can obviously be extended to all the periods individuated by the function $v_x(x) = v_0 \sin(kx)$

After the first expansion, gas will be at thermal equilibrium in each period of the initial velocity distribution $v_x(x) = v_0 \sin(kx)$, or at least its velocity distribution will be much more similar to Maxwellian one. Thus we cannot expect to observe a multi-stream flow after the conclusion of the first expansion if we have not already seen at least one before.

In conclusion, we have proved that the necessary and sufficient condition to observe at least one multi-stream flow is to observe it in planes $x = \frac{(2z+1)\pi}{k}$, $z \in \mathbb{Z}$, immediately after the initial instant.

2 Multi-stream flow around $x = \frac{(2z+1)\pi}{k}$, $z \in \mathbb{Z}$

In order to make calculation easier, we will use in this paragraph a sistem of reference having x -axis chosen so that the initial macroscopic velocity distribution is $v_x(x) = -v_0 \sin(kx)$. Moreover, like the preceeding paragraph, we will focus only in the first period $(-\frac{\pi}{k} \leq x \leq \frac{\pi}{k})$, with particles moving toward plane $x = 0$.

We have proved that, if we consider particles in a parallelepiped of height $\lambda \sim \frac{1}{nr^2} \ll \frac{1}{k}$ centered in $x = 0$, they are expected to collide for their first time in $x = 0$. Thus, in the follow we will consider constant the velocities of these particles. Moreover, since we are just looking for an order of magnitude, we can take the height of the parallelepiped to be 2λ rather than λ . Finally, the bigger are the differences of the speeds, the higher will be the probability for a multi-stream flow to arise. Thus, we can expect the multi-stream flow to arise when particles at distance λ reach the plane $x = \frac{\pi}{k}$.

Time needed by these last particles to reach the central plane is

$$\tau_1 = \frac{\lambda}{v_0 \sin(k\lambda)}$$

A particle in the parallelepiped coming from a position x will arrive in the central plane after a time τ_1 only if its initial speed is

$$v_i^x(x) = -\frac{x}{\tau_1} = -\frac{xv_0}{\lambda} \sin(k\lambda)$$

Since the macroscopic speed of a particle initially at x is given by $v_x = -v_0 \sin(kx)$, the number of particles between x and $x + dx$ having a speed between $v_i^x(x)$ and $v_i^x(x) + dv_i^x$, dx being proportional to dv_i^x , is given by Maxwell distribution and it is

$$dN(x) = c \exp \left(-\frac{m(v_i^x(x) + v_0 \sin(kx))^2}{2k_b T} \right) dx \quad (1)$$

where c is a constant independent from x .

Hypothesis under which this equation hold are satisfied only if we consider particles coming from the parallelepiped; however, we can suppose equation (1) to be valid also for distances from the central plane bigger than λ , for two reasons:

- since $v_i^x(x) = -\frac{xv_0}{\lambda} \sin(k\lambda)$, the term $(v_i^x(x) + v_0 \sin(kx))^2$ grows fastly when $|x| > \lambda$, thus the right-hand side of (1) will decrease very fastly for $|x| > \lambda$;
- since for $|x| > \lambda$ we have $|v_i^x(x)| > |v_i^x(\lambda)|$, these particles are actually not important in the determination of maxima and minima of the velocity distribution function in $x = 0$ between $-|v_i^x(\lambda)|$ and $|v_i^x(\lambda)|$.

Thus, we can consider (1) valid for all particles in the first period.

Since $k\lambda \sim 1$,

$$v_i^x(x) + v_0 \sin(kx) \sim -xv_0k + \frac{1}{6}xv_0k^3\lambda^2 + xv_0k - \frac{1}{6}x^3v_0k^3 = \frac{xv_0k^3}{6}(\lambda^2 - x^2)$$

holds. The stationary points of this function are in $x = \pm \frac{\sqrt{3}}{3}\lambda$, and thus the minima of the right-hand side of (1) are in these two values of x . For evident symmetry reasons, the distribution of velocities in $x = 0$ must be an even function; consequently, without loss of generality we will study it for $x \geq 0$. Equation

$$v_i^x(x) + v_0 \sin(kx) = 0$$

has got only two solutions, $x = 0$ and $x = \lambda$. These are two local maxima of the right-hand side of (1). We have already found a local minimum between them, in $x = \frac{\sqrt{3}}{3}\lambda$; thus, if

$$\frac{dN\left(\frac{\sqrt{3}}{3}\lambda\right)}{dN(0)} = \frac{1}{2}$$

we will have at least two local maxima of (1) that are twice the value of the local minimum between them.

From the last equation it follows that

$$\exp \left(-\frac{m \left(v_i^x \left(\frac{\sqrt{3}}{3}\lambda \right) + v_0 \sin \left(\frac{\sqrt{3}}{3}k\lambda \right) \right)^2}{2k_b T} \right) = \frac{1}{2}$$

or, since we are supposing $\lambda k \ll 1$,

$$\frac{\sqrt{3}\lambda v_0 k^3}{18} \left(\lambda^2 - \frac{1}{3}\lambda^2 \right) = \sqrt{2 \ln 2} \sqrt{\frac{k_b T}{m}} \Rightarrow v_0 = 9\sqrt{6 \ln 2} \frac{1}{k^3 \lambda^3} \sqrt{\frac{k_b T}{m}} \quad (2)$$

Since there is a linear function linking x and v_i^x , we can write the left-hand side of (1) as a function of v_i^x , $dN(v_i^x)$, which still has got at least two local maxima that are twice the value of the local minimum between them. However, $\frac{dN(v_i^x)}{dv_i^x}$ is (within a constant normalisation prefactor) the velocity distribution function in $x = 0$ at time τ_1 , since we have proved that the initial speed v_i^x of each particle is expected to be constant until it reaches $x = 0$; by definition, a multi-stream flow arises. Obviously, if we take values of v_0 greater than the limiting one found in (2), a multi-stream flow will still arise.

Thus, the condition for v_0 is (approximated to an order of magnitude)

$$v_0 \geq \frac{n^3 r^6}{k^3} \sqrt{\frac{k_b T}{m}}$$

Since we have supposed $k\lambda \ll 1 \Rightarrow nr^2 \gg k$, this result is consistent with the hypothesis $v_0 \gg \sqrt{\frac{k_b T}{m}}$. Moreover, in order to estimate λ we have approximated $v_x(x) \sim v_0 kx$ for $kx \ll 1$ and we have neglected random thermal oscillations; however, in §2 we have approximated $v_x(x)$ to the third order, and we have considered a Maxwell distribution. This is not completely legal. Despite this, as the local minimum of the velocity distribution function is just one half of the local maxima, we cannot choose a value of λ of an order of magnitude bigger than $\frac{1}{nr^2}$ if we want the particles in the parallelepiped to reach the central plane without colliding. Thus, the result found is also consistent with the definition of λ .

Finally, if we do not suppose $k\lambda \ll 1$, particles having a macroscopic speed of the same order of v_0 will reach $x = 0$ without collisions, thus they will keep their initial speed v_i^x . However, if $v_i^x(x) \gg \sqrt{\frac{k_b T}{m}}$ and $kx \sim 1$, we are sure a multi-stream flow will arise in $x = 0$ when these particles reach the central plane. Thus, if $k\lambda$ is of the same order of unity, we just have to suppose

$$v_0 \gg \sqrt{\frac{k_b T}{m}}$$