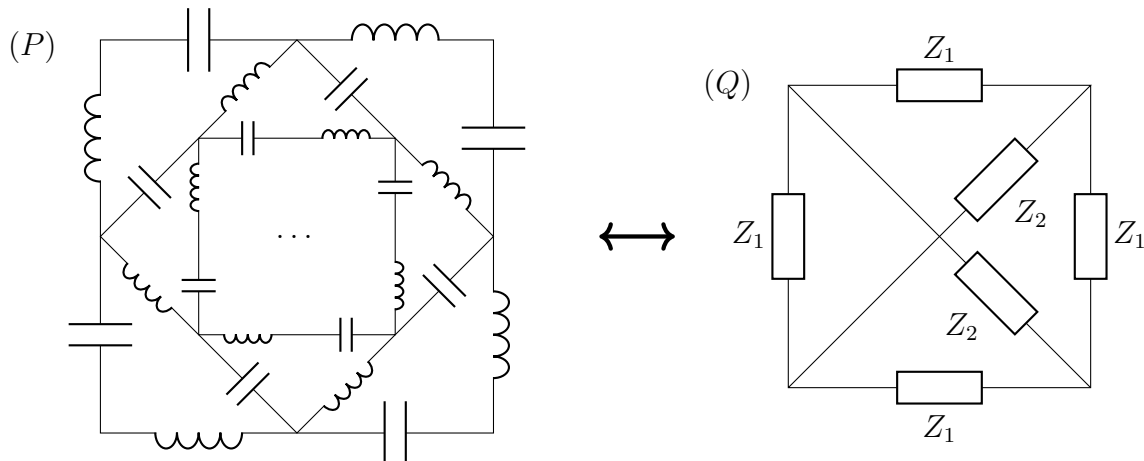
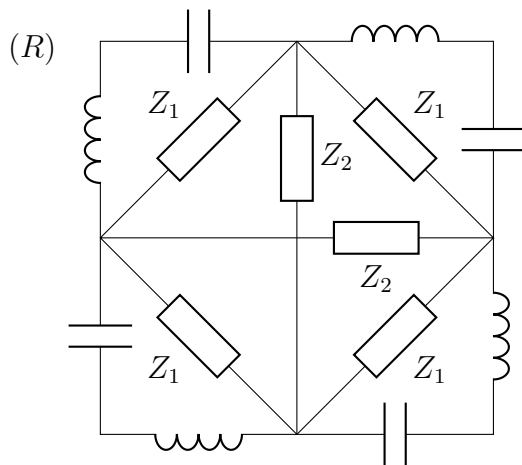


Problem 3

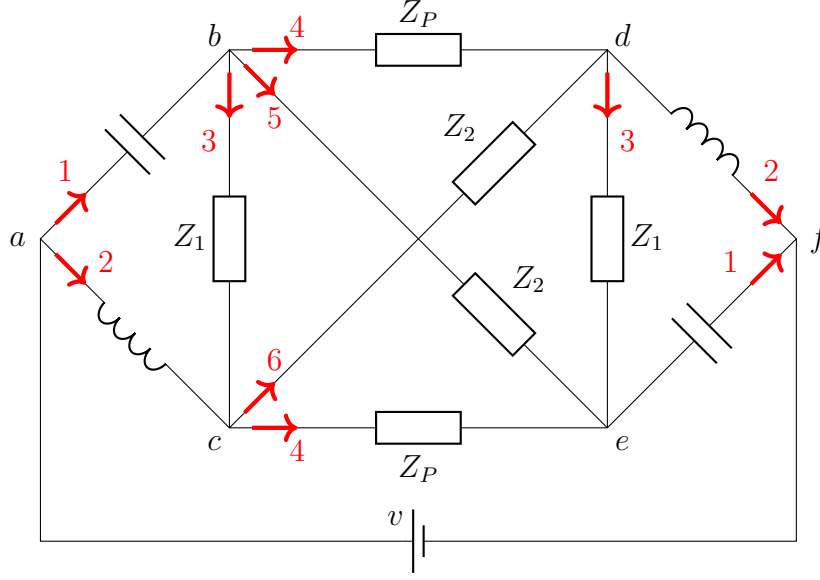
By linearity, at a given frequency ω the four-terminal infinite circuit (P) is equivalent to a square network (Q) of impedances. Due to $\pi/2$ rotational symmetry, we only need two impedances Z_1, Z_2 for the edges and diagonals respectively.



Then, (Q) is equivalent to the following circuit (R) :



Let Z be the impedance between opposite corners. For circuit (Q) , it is easy to see that $Z = \frac{1}{1/Z_1 + 1/Z_2}$. For circuit (R) , we have the following equivalent circuit, where $Z_P = \frac{1}{1/Z_1 + 1/(Z_C + Z_L)}$. Because a π rotation is equivalent to inverting the voltage and currents, the independent currents are reduced to $I_1, \dots, I_6$.



Applying the junction and loop rules, we have:

$$\begin{cases} I_5 = I_1 - I_3 - I_4 & \text{junction } b \\ I_6 = I_2 + I_3 - I_4 & \text{junction } c \\ Z_L I_2 + Z_P I_4 + Z_C I_1 = v & \text{loop } acef \\ Z_C I_1 + Z_1 I_3 - Z_L I_2 = 0 & \text{loop } abc \\ Z_1 I_3 + Z_2 I_6 - Z_P I_4 = 0 & \text{loop } bcd \\ Z_1 I_3 + Z_P I_4 - Z_2 I_5 = 0 & \text{loop } bce \end{cases}$$

$$\Rightarrow \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ Z_C & Z_L & 0 & Z_P & 0 & 0 \\ Z_C & -Z_L & Z_1 & 0 & 0 & 0 \\ 0 & 0 & Z_1 & -Z_P & 0 & Z_2 \\ 0 & 0 & Z_1 & Z_P & -Z_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow Z = \frac{v}{I_1 + I_2} = \frac{2(Z_1 + Z_2)Z_C Z_L + Z_1 Z_2(Z_C + Z_L)}{(Z_1 + Z_2)(Z_C + Z_L) + Z_1 Z_2} = \frac{2Z_C Z_L + Z(Z_C + Z_L)}{Z + Z_C + Z_L}$$

$$\Rightarrow Z^2 = 2Z_C Z_L = \frac{2L}{C}$$

The physically relevant value is $Z = \sqrt{\frac{2L}{C}}$. Interestingly, the infinite circuit behaves like a resistance $\sqrt{\frac{2L}{C}}$ at all frequencies, even though every finite circuit in the limiting process

has imaginary impedance. This shows that $I(t) = V_0 \sqrt{\frac{C}{2L}}$ for $t > 0$.