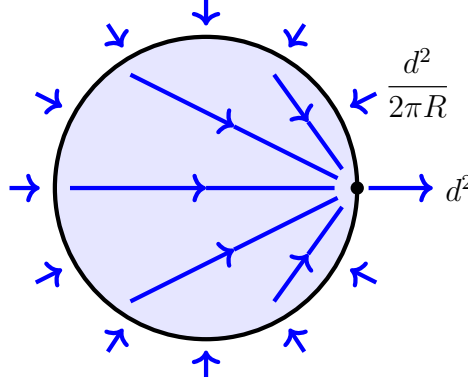


Problem 4

Let the circular region be C with boundary ∂C . Let the displacement at hole $0 \leq j < n = 6$ be x_j ; incompressibility implies $\sum x_j = 0$. Using the effective centrifugal potential $V = -\frac{1}{2}\Omega^2 r^2$, the change in potential energy is:

$$U = \sum \int_0^{x_j} -\frac{1}{2}\Omega^2(R + d + x)^2 \cdot \rho d^2 dx \approx -\frac{1}{2} \underbrace{\rho R \Omega^2 d^2}_k \left(R \underbrace{\sum_0 x_j}_0 + \sum x_j^2 \right)$$

Next, we'll find the kinetic energy. First, consider a situation with one hole at $R\hat{x}$ where water flows through ∂C uniformly to conserve volume. Then, taking $\dot{x} = 1$, the flow rate through the hole is d^2 and the flow rate per unit length through ∂C is $-\frac{d^2}{2\pi R}$. We claim that the solution is a radial flow $\vec{v}_0(\vec{r}) = -\frac{d}{\pi} \frac{\vec{r} - R\hat{x}}{|\vec{r} - R\hat{x}|^2}$ for $|\vec{r} - R\hat{x}| \gg d$. This is true because the angle that any portion of ∂C subtends from $R\hat{x}$ is half the angle subtended from $\vec{0}$.



Note that this is a potential flow satisfying $\vec{v}_0 = \nabla \phi_0$ where $\phi_0(\vec{r}) = -\frac{d}{\pi} \ln \left| \frac{\vec{r}}{R} - \hat{x} \right|$. Thus, we can use superposition to obtain the flow for all holes, $\vec{v}(\vec{r}) = \sum \dot{x}_j \vec{v}_j(\vec{r})$, where each $\vec{v}_j$ is an appropriately rotated copy of $\vec{v}_0$. The condition $\sum \dot{x}_j = 0$ means the flow through the boundary will be 0 as expected. Then, we obtain the kinetic energy:

$$T = \iint_C \frac{1}{2} \rho d |\vec{v}|^2 dA = \frac{1}{2} \left(\sum_j A_{jj} \dot{x}_j^2 + \sum_{j \neq k} \dot{x}_j \dot{x}_k \underbrace{\rho d \iint_C \vec{v}_j \cdot \vec{v}_k dA}_{A_{jk}} \right)$$

We do not write A_{jj} as an integral because the integrand goes as $1/r^2$ near the hole and the integral thus diverges. Physically, this means that behavior at scales $r \sim d$ is important. This is not an issue for A_{jk} , where the integrand goes as $1/r$.

Note that $A_{jk} = A_{kj}$, and that $A_{jk} = A_{j-k}$ depends only on $j - k$ by rotational symmetry. Then, considering the Lagrangian $L = T - U$, we have the Euler-Lagrange equations:

$$0 = \frac{\partial L}{\partial x_j} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{x}_j} = kx_j - \sum_k A_{jk} \ddot{x}_k = kx_j - (A\ddot{\vec{x}})_j \Rightarrow \ddot{\vec{x}} = kA^{-1}\vec{x}$$

Consider a mode $\vec{x}(t) = \vec{x}e^{ct}$ with doubling time $T = \ln(2)/c$. Note that $\vec{x}$ is an eigenvector of A with eigenvalue $\lambda = k/c^2$. On the other hand, because A is a circulant matrix, its eigenvectors are $\vec{x}_m$ with $x_{m,j} = e^{2\pi imj/n}$. Here, $m = 1, \dots, n-1$ satisfy the $\sum x_j = 0$ condition, while $m = 0$ does not. The eigenvalues are:

$$(A\vec{x}_m)_j = \sum_k A_{jk}x_{m,k} = e^{2\pi imj/n} \sum_k A_{j-k}e^{2\pi im(k-j)/n} \Rightarrow \lambda_m = \sum_j A_j e^{-2\pi imj/n}$$

It remains to find the A_j for $j \neq 0$. First, consider the following identity:

$$\iint_C \vec{v}_0 \cdot \vec{v}_j dA = \iint_C \nabla \phi_0 \cdot \vec{v}_j dA = \iint_C \nabla \cdot (\phi_0 \vec{v}_j) - \underbrace{\phi_0 \nabla \cdot \vec{v}_j}_0 dA = \oint_{\partial C} \phi_0 \vec{v}_j \cdot d\vec{n}$$

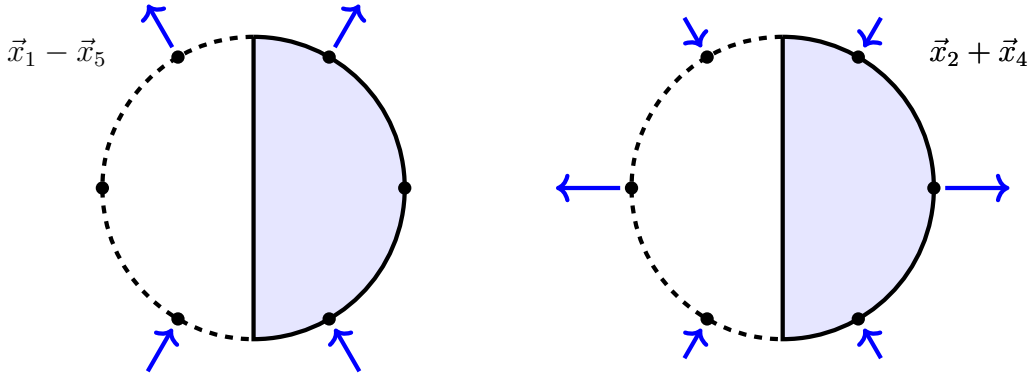
As before, we have $\vec{v}_j \cdot \hat{n} = -\frac{d}{2\pi R}$ far away from hole j . Thus, because $\nabla \cdot \vec{v}_j = 0$ everywhere in C , the flux near hole j must be positive to make the total flux 0. Parameterizing ∂C by the polar angle θ , we can thus approximate $\vec{v}_j \cdot \hat{n} = \frac{d}{R} \left(\delta\left(\theta - \frac{2\pi j}{n}\right) - \frac{1}{2\pi} \right)$. Furthermore,

$\phi_0 = -\frac{d}{\pi} \ln \left| \frac{\vec{r}}{R} - \hat{x} \right| = -\frac{d}{\pi} \ln |2 \sin(\theta/2)|$. This gives:

$$A_j = \rho d \int_0^{2\pi} \phi_0 \vec{v}_j \cdot \hat{n} R d\theta = \frac{\rho d^3}{\pi} \left(\underbrace{\frac{1}{2\pi} \int_0^{2\pi} \ln |2 \sin(\theta/2)| d\theta}_0 - \ln |2 \sin(\pi j/n)| \right)$$

In particular, $A_1 = A_5 = 0$, $A_2 = A_4 = -\frac{\ln(3)}{2\pi} \rho d^3$, and $A_3 = -\frac{\ln(2)}{\pi} \rho d^3$. Thus, we compute $\lambda_1 = \lambda_5 = A_0 + \frac{\ln(12)}{2\pi} \rho d^3$, $\lambda_2 = \lambda_4 = A_0 - \frac{\ln(4/3)}{2\pi} \rho d^3$, and $\lambda_3 = A_0 - \frac{\ln(3/2)}{\pi} \rho d^3$. The dominant mode in the full circle corresponds to the smallest eigenvalue, λ_3 .

For the half circle, taking hole 0 to be at the center, we can use the method of images: any solution x_0, x_1, x_5 for the half circle corresponds to one for the full circle with $x_2 = x_1, x_3 = x_0, x_4 = x_5$. Then, the modes of the half-circle are $\vec{x}_1 - \vec{x}_5$ and $\vec{x}_2 + \vec{x}_4$. Only $\vec{x}_2 + \vec{x}_4$ results in a nonzero amplitude at hole 0, so the desired eigenvalue for the half-circle is λ_2 .



$$\text{Finally, } T_2^2 - T_1^2 = \frac{\ln(2)^2}{k} (\lambda_2 - \lambda_3) = \boxed{\frac{\ln(2)^2 \ln(27/16)}{2\pi} \frac{d}{R\Omega^2}} \approx 0.04001 \frac{d}{R\Omega^2}.$$